LSB Dithering in MASH Delta–Sigma D/A Converters

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***Abstract*—Theoretical sufficient conditions are presented that ensure that the quantization noise from every constituent digital delta–sigma ($\Delta\Sigma$) modulator in a multistage digital $\Delta\Sigma$ modulator is asymptotically white and uncorrelated with the input. The conditions also determine if spectral shape can be imparted to the dither's contribution to the power spectral density of the multistage digital $\Delta\Sigma$ modulator's output. A large class of popular multistage digital $\Delta\Sigma$ modulators that satisfy the conditions are identified and tabulated for easy reference.**

***Index Terms*—Delta–sigma ($\Delta\Sigma$) modulation, dither techniques, MASH, quantization.**

I. Introduction

Digital-to-analog converters (DACs) and fractional-N phase-locked loops (PLLs) based on digital delta–sigma ($\Delta\Sigma$) modulators, collectively referred to as $\Delta\Sigma$ DACs, are enabling components in consumer communications and entertainment products including cellular telephones, wireless LANs, modems, and MP3 players. Digital $\Delta\Sigma$ modulators coarsely quantize over-sampled discrete-amplitude input signals within a feedback loop such that the power of the resulting quantization noise is suppressed (or *shaped*) within some frequency band of interest. Many popular $\Delta\Sigma$ DACs are based on an important class of $\Delta\Sigma$ modulators called MASH $\Delta\Sigma$ modulators: they comprise a cascade of single-quantizer digital $\Delta\Sigma$ modulators (SQDSMs) whose outputs are combined to achieve aggressively shaped overall quantization noise [1], [2].

In many, but not all SQDSM and MASH $\Delta\Sigma$ modulators, the *1-bit* (or *LSB*) *dithering* technique (adding a white 1-bit random sequence to the least significant bit (LSB) of the input of the $\Delta\Sigma$ modulator) can make the quantization noise asymptotically white and independent of the modulator's input [3]–[7]. These properties are crucial for the use of digital $\Delta\Sigma$ modulators in a $\Delta\Sigma$ DAC. Theoretical prior art has identified classes of SQDSMs in which 1-bit dithering offers the aforementioned benefits [8], [9].

However, in the case of MASH $\Delta\Sigma$ modulators, inadequate prior art has forced designers into an *ad hoc* application of the 1-bit dithering technique: some have added the dither to the input of only the last SQDSM in the cascade, others have added a different dither sequence to every SQDSM in the MASH system, others still have dithered only the input of the first SQDSM, all with varying degrees of success. The situation is compounded by the different forms of MASH $\Delta\Sigma$ modulators used in $\Delta\Sigma$ DAC applications: in some systems the individual SQDSM outputs are combined in digital domain (*digital combination MASH*) e.g., fractional-N PLLs [10]; whereas in others the SQDSM outputs are combined using error-prone analog circuitry (*analog combination MASH*) e.g., high resolution DACs [2] and noise cancellation fractional-N PLLs [7], [11]. Furthermore, the designers of high resolution MASH based DACs have used filtered dither (henceforth referred to as *shaped dithering*) to suppress the dither noise floor in the frequency band of interest.

This paper presents a theoretical analysis of the properties of the quantization noise in digital and analog combination MASH systems with a filtered 1-bit random sequence added to the input. Specifically, it presents sufficient conditions which, if satisfied by the MASH system and the dither filter, ensure that the quantization noise from every SQDSM is asymptotically white and independent of the MASH system's input. The conditions are an extension of the set of conditions that were presented in [8], [9]. The paper also presents a relaxed set of conditions that ensure that the quantization noise from only the final SQDSM is asymptotically white and independent of the MASH system's input; the final SQDSM's quantization noise is very important to the designers of digital combination MASH as explained later in this paper. Finally, the paper also identifies a class of popular MASH systems that satisfy the presented conditions resulting in a reference table for designers interested in employing 1-bit dithered MASH $\Delta\Sigma$ DACs.

The paper is organized as follows. Section II presents an overview of MASH digital–analog (D/A) conversion. Section III presents an overview of dithering in MASH systems. Section IV presents the theoretical analysis leading to the aforementioned sufficient conditions. Section V identifies popular MASH $\Delta\Sigma$ modulators that satisfy the conditions.

II. Overview of MASH D/A Conversion

A typical MASH $\Delta\Sigma$ D/A system is made of a cascade of SQDSMs: the input is quantized in steps—coarse quantization of the input, then finer quantization of the first SQDSM's quantization noise and so forth; the outputs of the individual SQDSMs are combined appropriately to form the final output signal. Fig. 1(a) shows a generic MASH system that is a cascade of K individual SQDSMs. The ith SQDSM has signal and noise transfer functions, $\mathrm{STF}_i(z)$ and $\mathrm{NTF}_i(z)$, respectively. It employs a midtread requantizer of step size N_i, where N_i is a positive integer, and forward transmission and feedback

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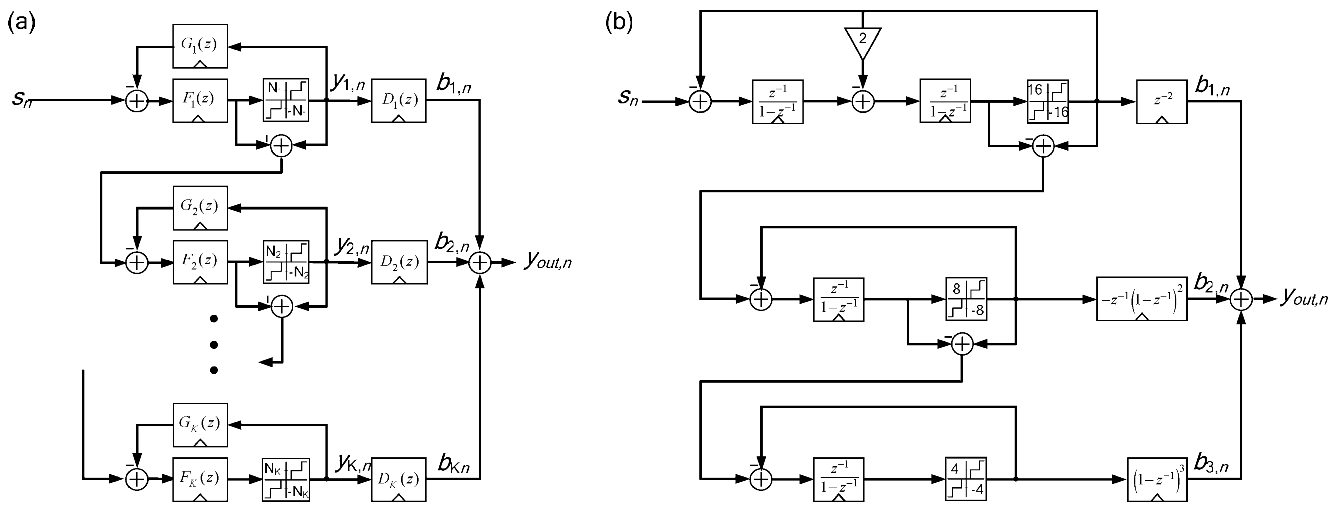

Fig. 1. (a) Gneric MASH digital $\Delta\Sigma$ modulator. (b) Example 2-1-1 MASH digital $\Delta\Sigma$ modulator.

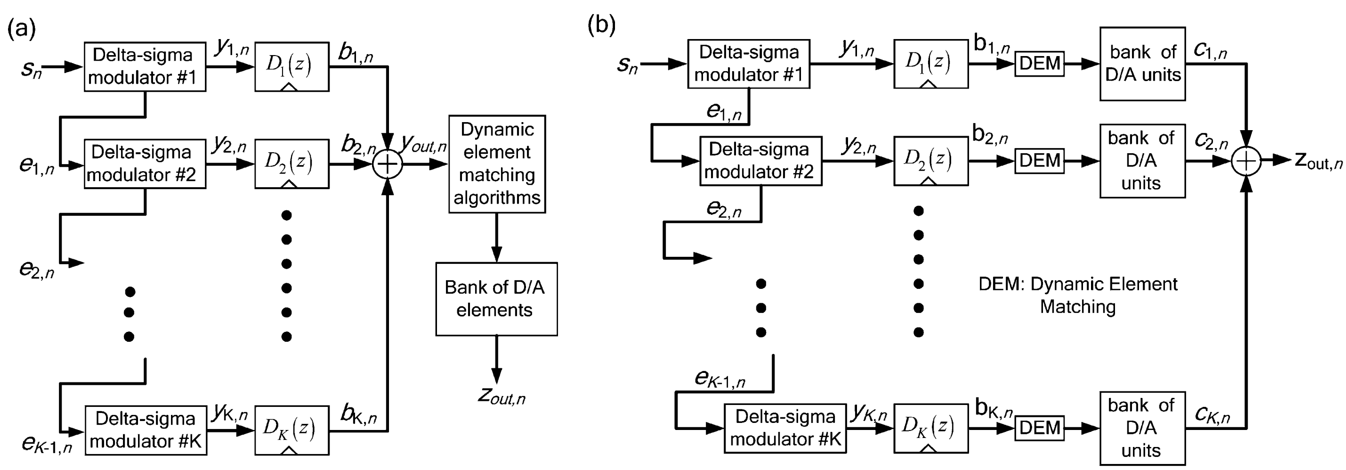

Fig. 2. (a) A generic digital combination MASH system (b) A generic analog combination MASH system

filters, $F_i(z)$ and $G_i(z)$. The midtread requantizer[1] internal to the ith SQDSM is henceforth referred to as the ith *requantizer*. The midtread requantizers in this paper do not overload unless otherwise specified. The quantization noise from the ith requantizer $e_{i,n}$, is computed by subtracting its input $r_{i,n}$ from its output $y_{i,n}$; it is fed as input to the succeeding i.e., the $(i+1)$th SQDSM as shown in the figure. The step-sizes of successive requantizers decrease in integer ratios to provide overall coarse quantization

$$N_j = m_{ji}N_i, \qquad m_{ji} \in \mathbb{Z}, m_{ji} > 0; \quad 1 \leqslant j < i \leqslant K. \quad (1)$$

The outputs $y_{i,n}$ are combined using a bank of post-processing filters $D_i(z), i = 1, 2, \ldots, K$, to produce a single output $y_{\text{out},n}$

$$y_{\text{out},n} = \sum_{i=1}^{K} y_{i,n} * d_{i,n} = \text{stf}_{\text{eq},n} * x_n + \text{ntf}_{\text{eq},n} * e_{K,n} \quad (2)$$

[1]The operation of a midtread requantizer is defined in [9].

where $d_{i,n}, \text{stf}_{\text{eq},n}$, and $\text{ntf}_{\text{eq},n}$ are the impulse responses of the ith post-processing filter, and the so-called overall signal and noise transfer functions, $\text{STF}_{\text{eq}}(z)$ and $\text{NTF}_{\text{eq}}(z)$, respectively

$$\text{NTF}_{\text{eq}}(z) = \prod_{i=1}^{K} \text{NTF}_i(z); \quad \text{STF}_{\text{eq}}(z) = \text{STF}_1(z)D_1(z). \quad (3)$$

Fig. 1(b) shows an example MASH system called the 2-1-1 MASH $K = 3, F_1(z) = z^{-2}(1-z^{-1})^{-2}, G_1(z) = 2z - 1, F_2(z) = F_3(z) = z^{-1}(1-z^{-1})^{-1}, G_2(z) = G_3(z) = 1, D_1(z) = z^{-2}, D_2(z) = z^{-1}(1-z^{-1})^{-2}$, and $D_3(z) = (1-z^{-1})^3$.

The MASH systems differ in whether the post-processing filter outputs, $b_{i,n}$, are summed in the digital domain or in the analog domain. In the digital combination MASH, the outputs $b_{i,n}$ are summed using digital logic and the digital output sequence, $y_{\text{out}}[n]$, is sent to a bank of DAC elements as shown in Fig. 2(a). Sophisticated dynamic element matching (DEM) algorithms exploit redundancies in the DAC banks to ensure that mismatches among these 1-bit DAC elements do not corrupt the desired signal [12]–[14]. In fact, appropriately chosen DEM algorithms ensure that the effect of the mismatches within the DAC banks can be modeled using a gain error δ, and an uncorrelated additive noise source err_n [15][2]

$$z_{\mathrm{out},n} = (1+\delta)y_{\mathrm{out},n} + \mathrm{err}_n. \tag{4}$$

Since the summation of $b_{i,n}$ to generate $y_{\mathrm{out},n}$ is performed using digital logic (see (2)), the overall quantization noise in the digital combination MASH depends on err_n and the quantization noise of only the final (Kth) SQDSM

$$z_{\mathrm{out},n} = (1+\delta)\mathrm{stf}_{\mathrm{eq},n} * x_n + (1+\delta)\mathrm{ntf}_{\mathrm{eq},n} * e_{K,n} + \mathrm{err}_n. \tag{5}$$

In contrast, the overall quantization noise in the analog combination MASH depends on the quantization noise of all the SQDSMs as shown below. In analog combination MASH, the $b_{i,n}$ are sent to individual banks of DAC elements whose analog outputs are summed to produce the analog equivalent of $y_{\mathrm{out},n}$ [16]–[18]. The DEM algorithms are employed prior to each bank of DACs as shown in Fig. 2(b). The individual banks of DAC elements contribute different gain errors, δ_i, and different uncorrelated additive error sources, $\mathrm{err}_{i,n}$

$$c_{i,n} = (1+\delta_i)b_{i,n} + \mathrm{err}_{i,n}. \tag{6}$$

Because of the unequal gain errors, the sum of $c_{i,n}$ contains potentially the quantization noise from every SQDSM

$$z_{\mathrm{out},n} = (1+\delta_{\mathrm{av}})\mathrm{stf}_{\mathrm{eq}} * x_n + \sum_{i=1}^{K} a_{i,n} * e_{i,n} + \sum_{i=1}^{K} \mathrm{err}_{i,n} \tag{7}$$

where δ_{av} is a constant and $a_{i,n}, i = 1, 2, \ldots, K$, are impulse responses, all of which are determined by $\mathrm{ntf}_{i,n}, d_{i,n}$, and the gain errors, δ_i. Note that on some occasions, the SQDSM outputs, $y_{i,n}$, are sent to individual DACs followed by post-processing filters implemented in the analog domain (e.g., as switched capacitor filters [2]). It can be shown that the output in such cases is also of the above form. The rest of the paper discusses the effects of 1-bit dithering on the ensemble and time-average statistics of the various $e_{i,n}$ in a MASH system.

Note that digital and analog combination MASH systems offer contrasting advantages and disadvantages. The former are simpler to implement with fewer analog blocks. However, they require a lot of hardware to implement the DEM algorithms: complexity of DEM algorithms can grow exponentially with the number of bits of $y_{\mathrm{out},n}$ e.g., a popular dynamic element matching algorithm called the *tree structured mismatch shaping* requires $(2^M - 1)$ digital *switching blocks*[3] [19] where M is the number of bits of $y_{\mathrm{out},n}$. Analog combination MASH systems employ multiple, smaller banks of DACs, so the DEM hardware requirements are relaxed. Consequently, while the digital combination MASH is popular in fractional-N PLL applications, the analog combination MASH is popular in high resolution D/A conversion applications such as high-fidelity audio and video DACs.

[2]Not all DEM algorithms guarantee that the DAC bank can be modeled using (4); the relevant necessary and sufficient conditions on DEM algorithms are discussed in [15].

[3]A *switching block* is the composite of all the logic gates required to implement the algorithm.

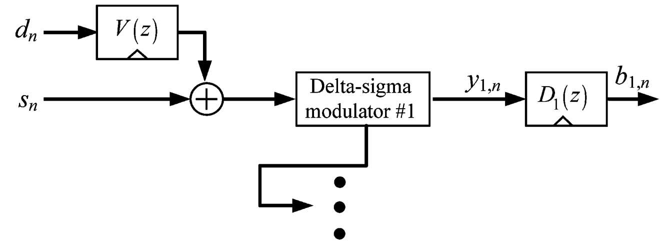

Fig. 3. LSB dithering in analog and digital combination MASH systems.

III. Overview of 1-Bit Dithering in MASH Systems

In 1-bit dithering of an analog or digital combination MASH system, a *1-bit dither sequence*, d_n, filtered by a shaping filter, $V(z)$, is added to the LSBs of a bounded *desired signal*, s_n, as shown in Fig. 3

$$x_n = s_n + v_n * d_n \tag{8}$$

The sum serves as input to the entire MASH system. The *1-bit dither sequence* is a random sequence with the following properties:

$$P(d_n = 0) = P(d_n = 1) = 0.5 \qquad \forall n \in \mathbb{Z},$$
$$d_n \text{ is independent of } s_l, d_m \qquad \forall l, m \in \mathbb{Z}, m \neq n. \tag{9}$$

Note that the shaping filter $V(z)$ is often not implemented explicitly. For example, $V(z) = (1 - z^{-1})$ could be implemented by simply adding d_n to the LSB of the first accumulator used in realizing the forward transmission filter, $F_1(z)$, of the first SQDSM (instead of adding it to the LSB of the input of the MASH); similarly, $V(z) = (1 - z^{-1})^2$ can be implemented by adding d_n directly to the LSB of the second accumulator of the first SQDSM. Experimental evidence suggests that for certain choices of $V(z), F_i(z), G_i(z)$, and N_i, every $e_{i,n}$ becomes a sequence whose samples are uniformly distributed and uncorrelated with x_n in a time-averaged sense

$$\lim_{L\to\infty} S_L(e_{i,n}) = M_{i,e} = 1/2, \quad \text{where } S_L(a_n) \triangleq \tfrac{1}{L}\textstyle\sum_{n=0}^{L-1} a_n \tag{10}$$

$$\lim_{L\to\infty} S_L((e_{i,n} - M_{i,e})x_{n-p}) = 0 \qquad \forall p \in \mathbb{Z} \tag{11}$$

$$\lim_{L\to\infty} S_L((e_{i,n} - M_{i,e})(e_{i,n-p} - M_{i,e})) = \sigma_{ii}^2 \delta[p] \quad \forall p \in \mathbb{Z}, \quad \text{where } \sigma_{ii}^2 = \tfrac{N_i^2 - 1}{12}. \tag{12}$$

Furthermore, it suggests that every pair of distinct quantization noise sequences, $e_{i,n}$ and $e_{j,n}$, is uncorrelated in a similar time-averaged sense

$$\lim_{L\to\infty} S_L((e_{i,n} - M_{i,e})(e_{j,n-p} - M_{j,e})) = 0 \qquad \forall i \neq j. \tag{13}$$

If the above properties were true, it follows from (7) that the power-spectral density (PSD) of the overall quantization noise of the corresponding digital and analog combination MASH systems has no spikes (other than those in the input)

$$\begin{aligned} S_{zz}(e^{j\omega}) = {} & |(1+\delta_{\mathrm{eq}})\mathrm{STF}_{\mathrm{eq}}(e^{j\omega})|^2 S_{ss}(e^{j\omega}) \\ & + |(1+\delta_{\mathrm{eq}})\mathrm{STF}_{\mathrm{eq}}(e^{j\omega})V(e^{j\omega})|^2 \sigma_d^2 \\ & + \sum_{i=1}^{K} |A_i(e^{j\omega})|^2 \sigma_{ii}^2 + \sum_{i=1}^{K} S_{\mathrm{DAC},i}(e^{j\omega}) \end{aligned} \tag{14}$$

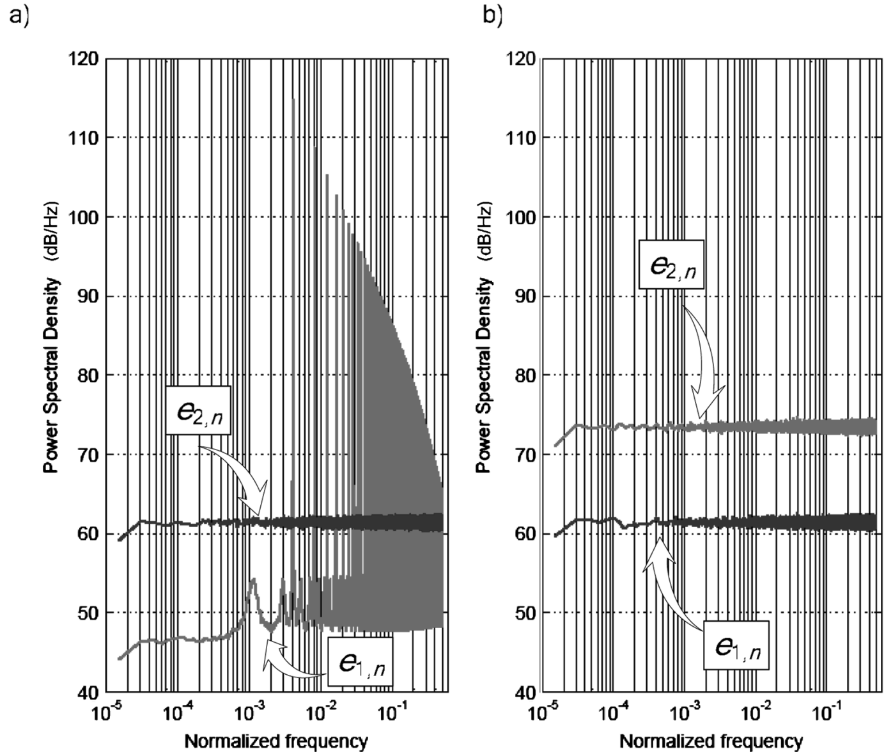

Fig. 4. (a) Effect of LSB dithering on the quantization noise in a 1-2 MASH system. (b) Effect of LSB dithering on the quantization noise in a 2-1 MASH system.

where $S_{ss}(e^{j\omega})$ is the PSD of $s_n, S_{\mathrm{DAC},i}(e^{j\omega})$ is the PSD of $\mathrm{err}_{i,n}, A_i(z)$ is the Z-transform[4] of $a_{i,n}, \sigma_{ii}^2$ is the variance of $e_{i,n}$, and σ_d^2 is the variance of the 1-bit dither, d_n.

However, (10)–(13) are not generally true as illustrated in Fig. 4. The figure shows the simulated PSD[5] of the quantization noise of the individual SQDSMs in two MASH systems for dc inputs, both with $V(z) = 1$. The plots in Fig. 4(a) correspond to $K = 2, F_1(z) = z^{-1}(1 - z^{-1})^{-1}, G_1(z) = 1, N_1 = 16384, F_2(z) = z^{-2}(1 - z^{-1})^{-2}, G_2(z) = 2z - 1$, and $N_2 = 4096$, a configuration popularly referred to as the "1–2" MASH. Notice that while $e_{1,n}$ has some spikes in its PSD, $e_{2,n}$ has none suggesting that $e_{1,n}$ does not possess one or more of the properties (10)–(12). Consequently, the "1–2" configuration is not desirable as an analog combination MASH system whereas it could be used as a digital combination MASH. The plots in Fig. 4(b) correspond to $K = 2, F_1(z) = z^{-2}(1 - z^{-1})^{-2}, G_1(z) = 2z - 1, N_1 = 16384, F_2(z) = z^{-1}(1 - z^{-1})^{-1}, G_2(z) = 1$, and $N_2 = 4096$, a configuration popularly referred to as the "2–1" MASH. Surprisingly, both $e_{1,n}$ and $e_{2,n}$ display a flat PSD without any spikes suggesting that the "2–1" configuration could be used as both a digital or an analog combination MASH system. Simulations of the above two MASH systems with other inputs e.g., sinusoidal inputs, also reveal that $e_{1,n}$ exhibits has spikes in the "1–2" configuration but not in the "2–1" configuration whereas $e_{2,n}$ exhibits no spikes in either case.

The properties of the quantization noise in a MASH system are also dependent on the shaping filter $V(z)$, as illustrated below. Note that the shaping filter $V(z)$, can be chosen to attenuate the dither noise floor that would have otherwise limited the signal-to-noise ratio (SNR) in the frequency band of interest. Fig. 5(a) shows the simulated PSDs (plotted against normalized angular frequencies) of the output $z_{\mathrm{out},n}$, of a digital combination MASH. The bank of DACs are assumed to be ideal without any errors; the MASH system has only one stage for the sake of simplicity: $K = 1, F_1(z) = z^{-3}(1 - z^{-1})^{-3}, G_1(z) = 3z^2 - 3z + 1$, and $N_1 = 16384$. The output PSDs are plotted for the cases of $V(z) = 1$ and $V(z) = 1 - z^{-1}$, respectively. In both cases, a sinusoidal input, s_n, was used. The low-frequency content of the overall quantization noise is dominated by the dither noise in both cases, but is significantly attenuated for $V(z) = (1 - z^{-1})$. Simulations confirmed similar behavior for other inputs, particularly for constant inputs as well. Without the shaping filter, i.e., for $V(z) = 1$, the dither noise floor can be made small only by increasing the number of bits representing the input of the MASH. However, this is wasteful in terms of the additional digital circuitry needed. Furthermore, in the case where the $\Delta\Sigma$ modulator is employed in a $\Delta\Sigma$ fractional-N PLL, the dither undergoes integration and appears in the phase of the PLL output [20], [21], potentially severely degrading the phase noise of the PLL output. As shown in the figure, the noise floor due to the dither is imparted a 20-dB/decade high-pass shape with $V(z) = (1 - z^{-1})$, without compromising the primary utility of the dither namely, imparting $e_{K,n}$ with the properties (10)–(12). If the band of interest in normalized frequencies is from 0 to α, it can be readily shown that $V(z) = (1 - z^{-1})$ improves the in-band SNR by

$$2\int_0^{\alpha} |V(e^{j\omega})|^2 d\omega.$$

[4]Assuming that the Z-transform exists.

[5]Simulated PSDs are plotted against normalized angular frequencies unless otherwise specified.

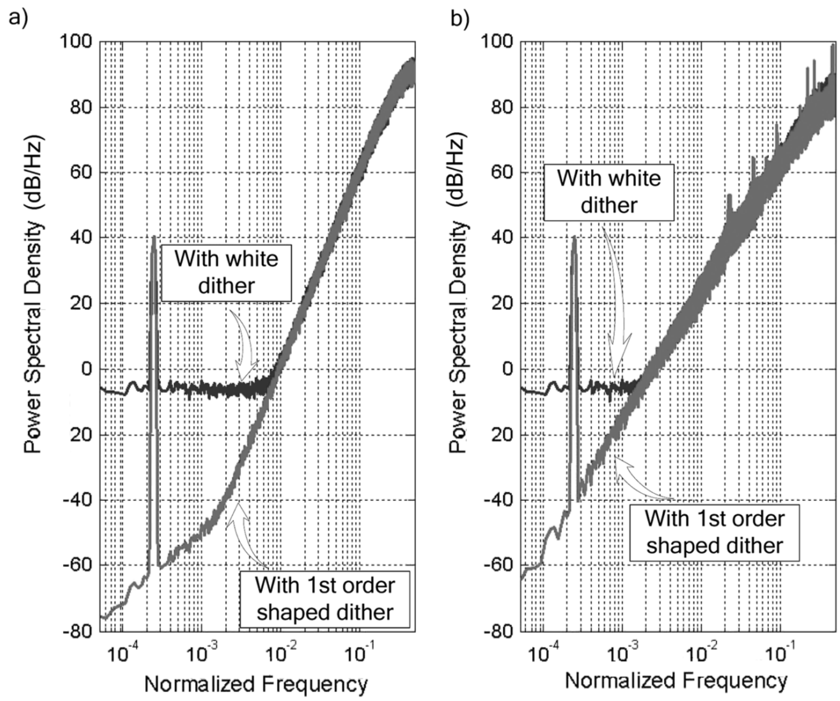

Fig. 5. (a) Effect of shaped dither on the output of a digital combination MASH with $F_1(z) = z^{-3}(1 - z^{-1})^{-3}$. (b) Effect of shaped dither on the output of a digital combination MASH with $F_1(z) = z^{-2}(1 - z^{-1})^{-2}$.

The advantages of shaped dithering have encouraged designers to aggressively shape the additive dither. For example, a 1-bit random sequence that has a third-order high-pass spectral shape has been added to the LSB of a third-order SQDSM in [6]. Unfortunately, arbitrarily shaped dither can not eliminate spurious tones in arbitrary MASH systems as illustrated in Fig. 5(b). The figure depicts the simulated PSD (plotted against normalized angular frequencies) of the output of a digital combination MASH in which $K = 1, F_1(z) = z^{-2}(1 - z^{-1})^{-2}, G_1(z) = 2z^2 - 1$, and $N_1 = 16384$ for the cases of $V(z) = 1$ and $V(z) = (1 - z^{-1})$. The evident spikes in the PSD indicate that $e_{K,n}$ i.e., $e_{1,n}$ in this case, does not possess at least some of the properties (10)–(12). Note that the first-order shaped dither is buried under the second-order quantization noise in the figure.

It is evident from the preceding discussion and Figs. 4 and 5 that $e_{i,n}$ possess the properties (10)–(13) only for certain MASH systems. The next section presents a theoretical analysis of the properties of $e_{i,n}$ leading to conditions for which (10)–(13) are true in probability. Specifically, a theorem is presented which specifies conditions on the impulse responses of $V(z)$ and $F_i(z)$ that are necessary and sufficient for the joint probability mass functions (pmf) of $e_{i,n}$ to acquire the following properties asymptotically.

- The pmf of $e_{i,n}$ converges to that of a uniform random variable as $n \to \infty$.
- The conditional pmf of $e_{i,n}$ given x_{n-p} converges to that of a uniform random variable as $n \to \infty$ for every finite p.
- The joint pmf of $e_{i,n}$ and $e_{i,n-p}$ converges to that of a pair of independent random variables as $n \to \infty$ for every finite $p \neq 0$.
- The joint pmf of $e_{i,n}$ and $e_{j,n-p}$ converges to that of a pair of independent random variables as $n \to \infty$ for $i \neq j$ and every finite integer p.

The results of the theorem are applicable to a very large class of input sequences s_n: they are applicable to constant inputs, deterministic nonconstant inputs such as quantized sinusoids, and more general classes of inputs as long as they are bounded integer sequences.

A corollary is also presented which stipulates that the same conditions on $V(z)$ and $F_i(z)$ are sufficient for (10)–(13) to be true in probability. A second theorem in Section V and an associated corollary present a relaxed set of conditions that are sufficient for only the quantization noise from the last SQDSM, $e_{K,n}$, to possess properties (10)–(12) in probability.

IV. Theoretical Results

It is assumed without loss of generality that all signals associated with the MASH system are zero for $n < 0$. Denote the input to the ith SQDSM as

$$x_{i,n} = e_{i-1,n} \qquad \forall 1 \leqslant i \leqslant K, \quad \text{where } e_{0,n} \triangleq s_n + v_n * d_n. \tag{15}$$

It can be shown [9] that the quantization noise of the ith SQDSM $(1 \leqslant i \leqslant K)$ is

$$e_{i,n} = N_i/2 - \langle x_{i,n} * f_{i,n} + N_i/2 \rangle_{N,i} \tag{16}$$

where $\langle x \rangle_{N,i} = x(\bmod N_i)$. Note that $e_{i,n}$ does not depend on the feedback filter of the ith SQDSM because the outputs of the feedback filters are integer multiples of N_i. The interested reader is referred to [9] for the details of this result. The theoretical analysis of the properties of $e_{i,n}$ proceeds by first expressing each $e_{i,n}$ directly in terms of the dither as shown in the following lemma.

Lemma 1: The quantization noise of the ith SQDSM is given by

$$e_{i,n} = N_i/2 - \langle z_{i,n} + N_i/2 \rangle_{N,i}$$
$$z_{i,n} = s_n * q_{i,n} + d_n * h_{i,n} \qquad \forall 1 \leqslant i \leqslant K \tag{17}$$

where $q_{i,n}$ and $h_{i,n}$ are impulse responses of filters $Q_i(z)$ and $H_i(z)$ that depend only on $V(z)$ and the various $F_i(z)$

$$Q_i(z) = (-1)^{i-1} \prod_{l=1}^{i} F_l(z)$$
$$H_i(z) = Q_i(z) \cdot V(z) \qquad \forall 1 \leqslant i \leqslant K. \tag{18}$$

Proof: See the Appendix. ■

The above lemma suggests that even though the dither is added to the input of the only the first SQDSM in the MASH, it effects the properties of every $e_{i,n}$. Furthermore, the properties of $e_{i,n}$ and $e_{j,n}$ depend on the impulse responses of the filters $H_i(z)$ and $H_j(z)$ namely, $h_{i,n}$ and $h_{j,n}$, respectively. The Theorem 1 presented below specifies conditions on $h_{i,n}$ and $h_{j,n}$ that are necessary and sufficient to ensure that $e_{i,n}$ and $e_{j,n}$ possess the ensemble statistical properties listed at the end of the previous section. The theorem is followed by two corollaries. The first corollary concerns the distribution of the values taken by a single instance of the quantization noise sequence e.g., $\{e_{i,0}, e_{i,1}, \ldots, e_{i,n}, \ldots\}$, an aspect that is of particular interest because of the time evolution of $z_{i,n}$. The proofs of the theorem and its corollary are presented in the Appendix.

Definition 1: The sequence e_n is said to be *asymptotically identically distributed and independent* of the sequence x_n if for every finite integer p,

$$P_{e_n \mid x_{n-p}}(a,b) \overset{n \to \infty}{\longrightarrow} P_U(a) \tag{19}$$

where $P_{A \mid B}(a,b)$ is the conditional probability of A given B, and U is a discrete random variable, integers a and b take values from the range of A and B, respectively.

Theorem 1: Suppose the input to a MASH system is $x_n = s_n + v_n * d_n$, where $s_n \in [-Q/2+1, Q/2]$ is an arbitrary bounded integer sequence and d_n is a *1-bit dither sequence*, and $h_{i,n}$ is the impulse response of $H_i(z) = (-1)^{i-1} V(z) \prod_{l=1}^{i} F_l(z)$ for all $1 \leqslant i \leqslant K$. Let $U_i, V_i, 1 \leqslant i \leqslant K$ be integer random variables that are uniformly distributed over $[-N_i/2+1, N_i/2]$ and such that V_j and U_i are pair-wise independent for all $1 \leqslant i,j \leqslant K$. Then, for $1 \leqslant i,j \leqslant K$ and $i \neq j$, the following are true.

i) $e_{i,n}$ is *asymptotically identically distributed and independent* of s_n if and only if at least one of the following conditions is true for each integer $p > 0$, and each integer $k, 0 < k < N_i$.
   1. The sequence $\langle k h_{i,r} \rangle_{N,i}$ does not converge to zero as $r \to \infty$;
   2. A nonnegative integer $s = s(p)$ exists such that $\langle k h_{i,s} \rangle_{N,i} = N_i/2$.

ii) $e_{i,n}$ is *asymptotically identically distributed and independent* of d_n if and only if at least one of the following conditions is true for each integer $p > 0$, and each integer $k, 0 < k < N_i$.

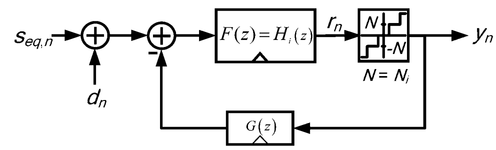

Fig. 6. Conceptual dithered SQDSM that generates the same quantization noise as the ith SQDSM in a MASH system.

   1. The sequence $\langle k h_{i,r} \rangle_{N,i}$ does not converge to zero as $r \to \infty$;
   2. A nonnegative integer $s = s(p) \neq p$ exists such that $\langle k h_{i,s} \rangle_{N,i} = N_i/2$.

iii) $(e_{i,n-p}, e_{i,n})$ converges in distribution to (V_i, U_i) for all integers $p \neq 0$ if and only if at least one of the following conditions is true for every integer $p > 0$, and each integer pair $(k_1, k_2) \neq (0,0)$ and $0 \leqslant k_1, k_2 < N_i$.
   1. The sequence $\langle k_1 h_{i,r} + k_2 h_{i,r+p} \rangle_{N,i}$ does not converge to zero as $r \to \infty$;
   2. A nonnegative integer $s = s(p) \neq p$ exists such that $\langle k_1 h_{i,s} + k_2 h_{i,s+p} \rangle_{N,i} = N_i/2$;
   3. A nonnegative integer $t = t(p) < p$ exists such that $\langle k_2 h_{i,t} \rangle_{Ni} = N_i/2$.

iv) $(e_{j,n-p}, e_{i,n})$ converges in distribution to (V_j, U_i) for all integers p if and only if at least one of the following conditions is true for every integer p and each integer pair $(k_1, k_2) \neq (0,0)$, such that $0 \leqslant k_1 < N_j$, and $0 \leqslant k_2 < N_i$.
   1. The sequence $\langle k_1 h_{j,r} + (N_j/N_i) k_2 h_{i,r+p} \rangle_{N,j}$ does not converge to zero as $r \to \infty$;
   2. A nonnegative integer $s = s(p)$ exists such that $\langle k_1 h_{j,r} + (N_j/N_i) k_2 h_{i,r+p} \rangle_{N,j} = N_j/2$;
   3. A nonnegative integer $t = t(p) < |p|$ exists such that $\langle k_2 h_{i,t} \rangle_{N,i}$ equals $N_i/2$ for $p > 0$ or $N_j/2$ for $p < 0$.

■

Remark 1: Note that $e_{i,n}$ (see (17) and (18)) is identical to the quantization noise e_n of an LSB dithered SQDSM shown in Fig. 6 when $N = N_i, F(z) = H_i(z)$, and $G(z)$ is an arbitrary filter with an integer valued impulse response; the input to the SQDSM is $s_{\mathrm{eq},n} = s_n * v_{\mathrm{inv},n}$ and $v_{\mathrm{inv},n}$ is the impulse response of $1/V(z)$. The properties of the quantization noise of the 1-bit dithered SQDSM have been studied in the authors' prior work (see Theorem 1 in [9]), for the case when $s_{\mathrm{eq},n}$ is bounded. The referred theorem presented almost identical (to those presented in parts (i), (ii), and (iii) in the above theorem) conditions on f_n that are necessary and sufficient for e_n to be *asymptotically identically distributed and independent* of $x_n = s_n + d_n$, and for (e_{n-p}, e_n) to converge in distribution to (V, U) where V and U are integer random variables that are uniformly distributed over $[-N/2+1, N/2]$. The above theorem extends the referred theorem in two crucial aspects. Foremost, part (iv) of the above theorem addresses the correlation between the quantization noise of multiple dithered SQDSMs namely, between $e_{i,n}$ and $e_{j,n}$ for $i \neq j$. Next, the theorem relaxes the condition that $s_{\mathrm{eq},n}$ be bounded; this condition is easily violated even if s_n were bounded e.g., when $V(z) = z^{-1}(1 - z^{-1})^{-1}$.

Remark 2: Note that the conditions of part (ii) imply those of part (i) since $s(p) \neq p$ in part (ii).

Remark 3: The conditions of part (iii) imply those of part (ii): conditions 1 and 2 of parts (i) and (ii) are, respectively, special cases of conditions 1 and 2 of part (iii) for $(k_1, k_2) = (k, 0) \neq (0, 0)$; since $(k, 0)$ does not satisfy the condition 3 of part (ii), the conditions of part (iii) imply those of part (ii).

Proof of Theorem 1: See the Appendix. ■

As mentioned before, the following corollary concerns the distribution of the values taken by a single instance of the quantization noise sequence, $\{e_{i,0}, e_{i,1}, \ldots, e_{i,n}, \ldots\}$: specifically, it claims that the conditions of Theorem 1 imply the time-averaged mean and correlation properties of $e_{i,n}$ (see (10)–(13)) in probability. The properties are reproduced below in the statement of the corollary for the sake of completeness and ease of reference.

Corollary:

i) The conditions of part (ii) of Theorem 1 are sufficient for the following to be true in probability:

$$\lim_{L\to\infty} S_L(e_{i,n}) = M_{i,e} = 1/2,$$
$$\lim_{L\to\infty} S_L((e_{i,n} - M_{i,e})s_{n-p}) = 0$$
$$\lim_{L\to\infty} S_L((e_{i,n} - M_{i,e})d_{n-p}) = 0 \qquad \forall p \in \mathbb{Z},$$
$$\text{where } S_L(x_n) = \frac{1}{L}\sum_{n=0}^{L-1} x_n. \quad (20)$$

ii) The conditions of part (iii) of Theorem 1 are sufficient for both (20) and the following to be true in probability:

$$\lim_{L\to\infty} S_L((e_{i,n} - M_{i,e})(e_{i,n-p} - M_{i,e})) = \sigma_i^2 \delta[p] \qquad \forall p \in \mathbb{Z},$$
$$\text{where } \sigma_i^2 = \frac{N_i^2 - 1}{12}. \quad (21)$$

iii) The conditions of part (iv) of Theorem 1 are sufficient for the following to be true in probability:

$$\lim_{L\to\infty} S_L((e_{i,n} - M_e)(e_{j,n-p} - M_e)) = 0 \qquad \forall p \in \mathbb{Z},$$
$$i \neq j. \quad (22)$$

Proof of Corollary: (i) and (ii): The proofs are similar to that of the corollary of Theorem 1 in [9] and hence are omitted here. Note that the conditions of part (ii) imply those of part (i). (iii): The proof is very similar to that of parts (i) and (ii). Note that the conditions of part (iii) also imply (20) because the conditions of part (ii) are implied by those of part (iii) as explained in Remark 2 following Theorem 1. ■

The above theorem and corollary can be used to analyze a large class of digital and analog combination MASH systems. For digital combination MASH systems, only $e_{K,n}$ is of concern: if $h_{K,n}$ satisfies the conditions of part (iii) of Theorem 1 then, $e_{K,n}$ possesses properties (20) and (21); if not, $e_{K,n}$ and hence the eventual analog output will most likely have undesirable artifacts such as spikes in the PSD. For analog combination MASH systems, all $e_{i,n}, 1 \leqslant i \leqslant K$ are of concern. So, every $h_{i,n}$ has to satisfy the conditions of part (iii) of Theorem 1 and furthermore, every pair of $h_{i,n}$ and $h_{j,n}, i \neq j$ have to satisfy the conditions of part (iv) of Theorem 1. While seemingly cumbersome, the conditions are easily verified for a large class of MASH systems as illustrated in the next section.

V. Dithering in Popular MASH D/A Systems

This section identifies classes of MASH systems that satisfy the conditions of Theorem 1; the results are summarized in Table I. The table will serve as a quick reference to determine if 1-bit dithering, shaped or otherwise, ensures that $e_{i,n}$ is asymptotically uniform, white, and uncorrelated with s_n and other $e_{j,n}$ in low-pass MASH systems i.e., systems for which the $\text{NTF}_{\text{eq}}(z)$ is ideally a high-pass filter. The table is divided into three sections: SQDSMs i.e., MASH systems with $K = 1$, digital combination MASH in which only $e_{K,n}$ is of concern, and analog combination MASH in which every $e_{i,n}$ is of interest. In each case, it is assumed that the impulse responses of the appropriate $G_i(z)$ are integer valued, the step-sizes are integer powers of 2 i.e., $N_i = 2^{M_i}$, and none of the quantizers overload. These assumptions are representative of most popular MASH systems. The numbers in the "$V(z) = (1 - z^{-1})^R$, allowed R" column indicate the highest order of high-pass shape that can be imparted to LSB dither and still ensure that $e_{i,n}$ has the aforementioned properties. An allowed R value of "none" in the digital combination MASH section implies that even unshaped dither i.e., $V(z) = 1$ does not ensure that $e_{K,n}$ has the aforementioned properties; in the analog combination MASH section it implies that at least one of the $e_{i,n}, 1 \leqslant i \leqslant K$ may not possess the properties. For instance, Table I suggests that a 1-1-1 configuration with LSB dither and $V(z) = (1 - z^{-1})$ can be used in a digital combination MASH but not in an analog combination MASH: either $e_{1,n}$ or $e_{2,n}$ may not possess the aforementioned properties and hence the eventual analog output might have undesirable artifacts such as spikes in the PSD or decreased in-band SNR.

The theoretical bases for the results in Table I are presented below: it is shown that the $H_i(z)$ for the various MASH systems shown in Table I satisfy the conditions of Theorem 1. Theorem 2 and its corollary concern digital combination MASH systems in which only $e_{K,n}$ is of interest; Theorem 3 concerns analog combination MASH systems in which the properties of every $e_{i,n}$, and every pair of $(e_{i,n}, e_{j,n})$ are of interest. Note that analogous results can be obtained for bandpass MASH systems i.e., systems for which $\text{NTF}_{\text{eq}}(z)$ is ideally a band stop filter, through a similar application of Theorem 1.

Theorem 2: Suppose a digital combination MASH with $H_K(z) = cz^{-X}(1 - z^{-1})^{-L}$, where $L > 1, c = \pm 1$ and $X \geqslant 0$ are integers, the impulse responses of $G_i(z)$ are integer valued, the step-sizes are integer powers of 2, $N_i = 2^{M_i}$ such that $N_i \geqslant N_j$ for all $i \leqslant j$, and none of the quantizers overload. Then, the quantization noise $e_{K,n}$ has the properties (20) and (21).

Proof: The MASH system can be redrawn with an appropriate integer scaling constant and delay elements at the output such that $H_K(z) = z^{-L}(1 - z^{-1})^{-L}$: the statistics of $e_{K,n}$ remain unchanged. Since only $e_{K,n}$ is of interest, corollary 1 can be used with $i = K$ to prove the result: it needs to be shown that $H_K(z) = z^{-L}(1 - z^{-1})^{-L}$ satisfies the conditions of part (iii) of Theorem 1. The results in [9] (see Theorems 2 and 3) prove the same. ■

Note that since only $e_{K,n}$ is of concern in a digital recombination MASH, the condition that none of the quantizers should

TABLE I
1-Bit Dithering in General Low-Pass MASH Systems

Single Quantizer ΔΣ Modulators		
Name of ΔΣ modulator	Forward transmission filter, $F(z)$	$V(z)=(1-z^{-1})^R$, acceptable R
1st order low pass	$z^{-1}(1-z^{-1})^{-1}$	None
L^{th} order low pass, $L>1$	$z^{-L}(1-z^{-1})^{-L}$	$L-2$
Digital Combination MASH Systems		
Type of MASH	$F_1(z)\cdot F_2(z)\cdot\ldots\cdot F_K(z)$	$V(z)=(1-z^{-1})^R$, allowed R
1-1 low pass	$z^{-2}(1-z^{-1})^{-2}$	0
1-1-1, 1-2, 2-1 low pass	$z^{-3}(1-z^{-1})^{-3}$	1
$m_1-m_2-\ldots-m_K$, $m_1+\ldots+m_K=L>4$ low pass	$z^{-L}(1-z^{-1})^{-L}$	$L-2$
Analog Combination MASH Systems		
Type of MASH	$F_1(z)\cdot F_2(z)\cdot\ldots\cdot F_K(z)$	$V(z)=(1-z^{-1})^R$, allowed R
1-1 low pass	$z^{-2}(1-z^{-1})^{-2}$	None
1-1-1, 1-2 low pass	$z^{-3}(1-z^{-1})^{-3}$	None
2-1 low pass	$z^{-3}(1-z^{-1})^{-3}$	0
1-a-b-c-… low pass	$z^{-(1+a+b+\ldots)}(1-z^{-1})^{-(1+a+b+\ldots)}$	None
L-a-b-c-…, $L>1$ low pass	$z^{-(L+a+b+\ldots)}(1-z^{-1})^{-(L+a+b+\ldots)}$	$L-1$

overload can be relaxed: it is only required that the Kth quantizer not overload as shown in the following corollary to Theorem 2. Of particular interest are those MASH $\Delta\Sigma$ modulators in which all but the final digital requantizer is a simple, single-bit requantizer.

Corollary to Theorem 2: Suppose that only the Kth quantizer in the digital combination MASH system in Theorem 2 does not overload. Even then, the quantization noise $e_{K,n}$ possesses the properties (20) and (21).

Proof: While $e_{K,n}$ is still given by (15), the other $e_{i,n}$ are different because they are allowed to overload. Specifically, $e_{i,n}$ are not bounded to be within $[-N_i/2+1, N_i/2]$; it can be shown that

$$e_{i,n} = N_i/2 - \langle e_{i-1,n} * f_{i,n} + N_i/2\rangle_{N,i} + l_{i,n}N_i \quad \forall 1 \leqslant i < K, \quad \text{where } l_{i,n} \in \mathbb{Z}. \tag{23}$$

Note the additional term $l_{i,n}N_i$ in (23) compared to (16). The additional term does not alter the relation between $e_{i,n}$ and $e_{i-1,n}$ derived in Lemma A1. Consequently, Lemma A1 still remains applicable; the result follows from Theorem 1 and its corollary. ■

Theorem 2 and its corollary prove the results summarized in the SQDSM and digital combination MASH sections of Table I. For instance, consider a digital combination MASH with $K=2, V(z)=(1-z^{-1}), F_1(z)=z^{-1}(1-z^{-1})^{-1}$, and $F_2(z)=z^{-2}(1-z^{-1})^{-2}$: since $H_K(z)=z^{-2}(1-z^{-1})^{-3}$, it follows from Theorem 2 that $e_{K,n}$ has the properties (20) and (21). For the same reason, the e_n of an SQDSM (i.e., a digital combination MASH with $K=1$) with $V(z)=(1-z^{-1})$ and $F(z)=z^{-3}(1-z^{-1})^{-3}$ possesses these properties as well. The SQDSMs with $F(z)=z^{-3}(1-z^{-1})^{-3}$ typically provide a third-order high-pass noise transfer function and are popular choices with $\Delta\Sigma$ fractional-N PLLs. By using 1-bit dither with a first-order high-pass shaping i.e., $V(z)=(1-z^{-1})$, this particular SQDSM simultaneously removes spikes from the SQDSM's overall quantization noise and ensures that the in-band phase noise is small.

In the case of analog combination MASH systems, it is desirable that every pair of $e_{i,n}$ and $e_{j,n}, i\neq j$ possess the property (22) apart from every $e_{i,n}$ possessing the properties (20) and (21). For a given analog combination MASH system, the corollary to Theorem 1 suggests that the conditions of both parts (iii) and (iv) of Theorem 1 are sufficient for these properties. Consequently, it needs to be verified that (a) every $h_{i,n}$ (the impulse response of $H_i(z)$) satisfies the conditions of part (iii) of Theorem 1, and (b) every pair of $h_{i,n}$ and $h_{j,n}$ satisfy the conditions of part (iv) of Theorem 1. The following Theorem 3 proves the same for a large class of $H_i(z)$: first, Lemma 2 proves that every pair of impulse responses $h_{i,n}$ and $h_{j,n}$ of this class of $H_i(z)$ satisfy the conditions of part (iv) of Theorem 1; then the lemma is used to prove Theorem 3.

Lemma 2: Suppose W and T are positive integers such that W/T is an integer, and $w_{x,n}$ and $t_{y,n}$ are the impulse responses of filters

$$W_X(z) = az^{-c}(1-z^{-1})^{-X}$$
$$T_Y(z) = bz^{-d}(1-z^{-1})^{-Y}, \qquad a,b \in \pm 1 \tag{24}$$

where $X > Y > 0$, and $c, d \geqslant 0$ are integers. Then, for every integer p and each integer pair $(k_1, k_2) \neq (0,0)$ such that $0 \leqslant k_1 < W$ and $0 \leqslant k_2 < T$, the sequence

$$\langle k_1 w_{X,r} + (W/T)k_2 t_{Y,r+p}\rangle_W$$

does not converge to zero as $r \to \infty$.

Proof: See the Appendix. ■

Theorem 3: Consider a dithered analog combination MASH system in which

$$V(z) = (1-z^{-1})^R, F_i(z) = z^{-X_i}(1-z^{-1})^{-L_i}$$
$$\text{and } N_i = 2^{M_i} \text{ where } R, L_i, X_i \geqslant 0, M_i \geqslant M_{i-1},$$

and $L_1 \geqslant 2 + R$, the impulse responses of $G_i(z)$ are integer valued and none of the quantizers overload. Then, every $e_{i,n}$ possesses properties (20) and (21), and every pair of $e_{i,n}$ and $e_{j,n}$ where $i \neq j$ has the property (22).

Proof: Note that

$$H_i(z) = (-1)^{i-1}V(z)\prod_{l=1}^{i} F_l(z)$$
$$= (-1)^{i-1}z^{-\sum_{l=1}^{i} X_l}(1-z^{-1})^{R-\sum_{l=1}^{i} L_l} \tag{25}$$

Since $L_1 \geqslant 2 + R$, it follows that $\sum_{l=1}^{i} L_l - R \geqslant 2$ for all $1 \leqslant i \leqslant K$. Therefore, Theorem 2 implies that every $e_{i,n}$ possesses the properties (20) and (21). Setting $w_n = h_{i,n}, t_n = h_{j,n}, N_H = N_i, N_T = N_j, X = R - \sum_{l=1}^{i} L_l, Y = R - \sum_{l=1}^{j} L_l, c = X - R$, and $d = Y - R$ in Lemma 2 proves that the impulse responses of $h_{i,n}$ and $h_{j,n}$ satisfy the first condition of part (iv) of Theorem 1. Note that $X < Y$ as required by Theorem 2. Therefore, every pair of $e_{i,n}$ and $e_{j,n}$ where $i \neq j$ has the property (22). ■

Theorem 3 proves the results summarized in the analog combination MASH section of Table I. For instance, consider an analog combination MASH with $K = 2, V(z) = (1-z^{-1}), F_1(z) = z^{-1}(1-z^{-1})^{-1}$, and $F_2(z) = z^{-2}(1-z^{-1})^{-2}$. Note that the same configuration was considered for a digital combination MASH in the discussion following Theorem 2: it was shown that since $H_K(z) = z^{-2}(1-z^{-1})^{-3}, e_{K,n}$ has the properties (20) and (21).

VI. Conclusion

A theoretical analysis of two classes of LSB dithered MASH systems is presented; a set of sufficient conditions that ensure that the quantization noise from every constituent SQDSM is asymptotically white and uncorrelated with the MASH system's input. A large class of popular analog and digital combination MASH systems have been shown to satisfy these sufficient conditions.

Appendix

Proof of Lemma 1: It follows from (15) and (16) that

$$e_{i-1,n} * f_{i,n} = m_{i,n}N_i - e_{i,n}, \quad \text{where } m_{i,n} \in \mathbb{Z}. \tag{26}$$

Similarly,

$$e_{i-2,n} * f_{i-1,n} = m_{i-1,n}N_{i-1} - e_{i-1,n}, \quad \text{where } m_{i-1,n} \in \mathbb{Z}. \tag{27}$$

Substituting (27) in (26) results in

$$e_{i,n} = m_{i,n}N_i - f_{i,n} * e_{i-1,n}$$
$$= m_{i,n}N_i - f_{i,n} * (m_{i-1,n}N_{i-1} - f_{i-1,n} * e_{i-1,n}),$$
$$m_{i,n}, m_{i-1,n} \in \mathbb{Z}.$$

Proceeding recursively and noting that $N_1, N_2, \ldots, N_{i-1}$ are positive integer multiples of N_i (see (1)) results in

$$e_{i,n} = m_n N_i + (-1)^i f_{i,n} * f_{i-1,n} * e_{0,n}$$
$$= m_n N_i + q_{i,n} * e_{0,n}, \quad \text{where } m_n \in \mathbb{Z}. \tag{28}$$

Substituting (28) and (15) in (16) results in (17) and (18). ■

Proof of Theorem 1: As mentioned before, Theorem 1 is an extension of Theorem 1 presented in [9]; their respective proofs are very similar. The quantization noise of the ith SQDSM (see (17)) can be written as

$$e_{i,n} = N_i/2 - \langle z_{i,n} + N_i/2\rangle_{N,i}, \quad \text{where } z_{i,n} = a_{i,n} + d_n * h_{i,n};$$
$$a_{i,n} = s_n * q_{i,n} \tag{29}$$

where as the quantization noise that the Theorem 1 in [9] concerns with is

$$e_n = N/2 - \langle z_n + N/2\rangle_N,$$
$$\text{where } z_n = a_n + d_n * f_n \text{ and } a_n = s_n * f_n. \tag{30}$$

For a fixed i, while the similarities are evident from (29) and (30), the differences are two-fold: first, in (29) filtering undergone by s_n is not the same as that undergone by d_n; second, unlike in [9], the conditional pmf of e_n given s_{n-p} and that of e_n given d_{n-p} are of interest instead of the conditional pmf of e_n given x_{n-p}. However, the proof of Theorem 1 in [9] does not rely on the form of a_n. Consequently, much of the proof of Theorem 1 follows from the proof of Theorem 1 in [9]. In fact, the conditions on f_n presented in Theorem 1 in [9] apply here with minor changes to $h_{i,n}$. The variations that concern the conditional pmf of e_n given s_{n-p} and that of e_n given d_{n-p} are pointed out below. Note that the joint pmf of $(e_i, e_{j,n-p})$ for $i \neq j$ is also of interest here and is considered separately.

Proof of Part (iv): The proof is very similar to the proof of part (ii) of Theorem 1 in [9]. Each of $e_{j,n-p}, e_{i,n}, V_j$, and U_i are bounded discrete random variables, so the convergence of $(e_{j,n-p}, e_{i,n})$ in distribution to (V_j, U_i) as $n \to \infty$ is equivalent to

$$P_{e_{j,n-p},e_{i,n}}(a,b) \stackrel{n\to\infty}{\longrightarrow} P_{V_j,U_i}(a,b) hfill$$
$$\forall a \in \mathbb{Z} \cap [-N_j/2+1, N_j/2]; \quad b \in \mathbb{Z} \cap [-N_i/2+1, N_i/2]. \tag{31}$$

It can be shown that the $N_j\, N_i$ uniform samples of the joint characteristic function (jcf) of two discrete random variables that take on values from finite sets of size N_j and N_i, and their joint pmf form a unique two-dimensional discrete Fourier transform pair: [9] argued the same for $N_j = N_i = N$ but the argument applies otherwise too. Consequently, (31) is equivalent to the following convergence:

$$\Phi_{e_{j,n-p}, e_{i,n}}\left(\frac{2\pi k_1}{N_j}, \frac{2\pi k_2}{N_i}\right) \overset{n\to\infty}{\longrightarrow} \Phi_{V_j,U_i}\left(\frac{2\pi k_1}{N_j}, \frac{2\pi k_2}{N_i}\right) = \delta[k_1]\delta[k_2] \quad \forall 0 \leqslant k_1 < N_j, \quad 0 \leqslant k_2 < N_i. \tag{32}$$

Note that the samples of the jcf of (V_j, U_i) simplify to a product of Kronecker delta functions because V_j and U_i are independent and uniformly distributed. Also note that the equivalence of (31) and (32) is just a special case (for discrete, finite range) of the continuity theorem (Theorem 26.3, pg. 359 in [22]). It can also be shown (see [9], or (17) in [23], or Lemma A1 in [24]) that if $e_{i,n}$ and $z_{i,n}$ are related by the fractional operator as shown in (29), then

$$\Phi_{e_{j,n-p},e_{i,n}}\left(\frac{2\pi k_1}{N_j}, \frac{2\pi k_2}{N_i}\right) = \Phi_{z_{j,n-p},z_{i,n}}\left(\frac{-2\pi k_1}{N_j}, \frac{-2\pi k_2}{N_i}\right) \quad \forall 0 \leqslant k_1 < N_j, \quad 0 \leqslant k_2 < N_i \tag{33}$$

By assumption, the samples of d_n are independent, identically distributed random variables also independent of s_n. Therefore, it follows from the properties of d_n and (29) that

$$\Phi_{z_{j,n-p}, z_{i,n}}(u,v) = \prod_{m=0}^{\min(n-p,n)} \Phi_d(uh_{j,n-p-m} + vh_{i,n-m}) \cdot \Phi_{a_{j,n-p}, a_{i,n}}(u,v) \cdot \begin{cases} \prod_{l=n-p+1}^{n} \Phi_d(vh_{i,n-l}), & \forall p > 0 \\ 1, & \forall p = 0 \\ \prod_{l=n+1}^{n-p} \Phi_d(uh_{j,n-p-l}), & \forall p < 0 \end{cases} \tag{34}$$

where $\Phi_{a_{j,n-p},a_{i,n}}(u,v)$ is the jcf of $(a_{j,n-p}, a_{i,n})$. Substituting $r = n-p-m$ in the first product in the RHS of (34), $r = n-l$ in the second product, and $u = -2\pi k_1/N_j, v = -2\pi k_2/N_i$, and substituting the result in (33) gives

$$\Phi_{e_{j,n-p}, e_{i,n}}\left(\frac{2\pi k_1}{N_j}, \frac{2\pi k_2}{N_i}\right) = \Phi_{a_{j,n-p}, a_{i,n}}\left(\frac{-2\pi k_1}{N_j}, \frac{-2\pi k_2}{N_i}\right) \cdot \prod_{r=0}^{n-\min(p,0)} \Phi_d\left(\frac{-2\pi k_1}{N_j}\left\{h_{j,r} + \frac{N_j}{N_i}h_{i,r+p}\right\}\right) \cdot \begin{cases} \prod_{r=0}^{p-1} \Phi_d(-2\pi k_2 h_{i,r}/N_i), & \forall p > 0 \\ 1, & \forall p = 0 \\ \prod_{r=0}^{-p-1} \Phi_d(-2\pi k_1 h_{j,r}/N_j), & \forall p < 0 \end{cases} \tag{35}$$

Both the RHS of (35) and the RHS of (32) equal unity for $k_1 = k_2 = 0$. Therefore, it remains to be shown that the RHS of (35) converges to zero as $n \to \infty$ for every integer p and each integer pair $(k_1, k_2) \neq (0,0)$ such that $0 \leqslant k_1 < N_j$ and $0 \leqslant k_2 < N_i$, if and only if at least one of the conditions of part (iv) is true. But first, note that the characteristic function of each d_n is

$$\Phi_{d_n}(u) = \Phi_d(u) = e^{ju*0}P_d(0) + e^{ju*1}P_d(1) = 0.5(1+e^{ju}) = e^{ju/2}\cos(u/2). \tag{36}$$

Sufficiency: It can be shown (see proof of Theorem 1 in [9]) that if conditions 2 or 3 (of part (iv)) are true, then one of the constituent $\Phi_d(\,\cdot\,)$ terms in the RHS of (35) equals zero: in such cases, the argument of the $\Phi_d(\,\cdot\,)$ term becomes an odd integer multiple of $\pi/2$. It can also be shown (see proof of Theorem 1 in [9]) that if condition 1 is true, then the infinite product in the RHS converges to zero as $n \to \infty$ for any integer p. The details of both arguments are explained in [9] and hence omitted here.

Necessity: Note that the first term in the RHS of (35) depends on s_n but not on d_n. Consequently, the RHS of (35) converges to zero as $n \to \infty$ for arbitrary s_n only if at least one of the $\Phi_d(\,\cdot\,)$ terms equals zero or if the infinite product converges to zero as $n \to \infty$. Therefore, the conditions of part (iv) are necessary.

Proof of Part (iii): The proof is similar to that of part (iv) with $i = j$. An almost identical result has been proved in part (ii) of Theorem 1 in [9].

Proof of Parts (i) and (ii): The proof is similar to that of part (iv) except that conditional pmfs and conditional characteristic functions are used instead of joint pmfs and joint characteristic functions: for e.g., proving that

$$P_{e_{i,n}\,|\,d_{n-p}}(a,b) \overset{n\to\infty}{\longrightarrow} P_{U_i}(a) \tag{37}$$

is equivalent to proving that

$$\Phi_{e_{i,n}\,|\,d_{n-p}}\left(\frac{2\pi k}{N_i}\right) \overset{n\to\infty}{\longrightarrow} \Phi_{U_i}\left(\frac{2\pi k}{N_i}\right) = \delta[k], \quad 0 \leqslant k < N_i \tag{38}$$

where $\Phi_{A\,|\,B}(u) = E(e^{juA}\,|\,B)$ is the conditional characteristic function of A given B, and $\delta[k]$ is the characteristic function of the uniform discrete random variable U_i. Note that the LHS of (38) is actually a function of both k and the value of $d_{n-p}(=b)$; however, the latter argument is suppressed for the sake of simplicity, and is shown explicitly only when required. Proceeding as before, it can be shown that

$$\Phi_{e_{i,n}\,|\,d_{n-p}}\left(\frac{2\pi k}{N_i}\right) = \Phi_{b_{i,n}\,|\,d_{n-p}}\left(\frac{-2\pi k}{N}\right) \cdot e^{\frac{-j2\pi k h_{i,p} d_{n-p}}{N_i}} \cdot \prod_{r=0, r\neq p}^{n-p} \Phi_d\left(\frac{-2\pi k h_{i,r}}{N_i}\right) \tag{39}$$

where $b_{i,n} = a_{i,n} - h_{i,p}d_{n-p}$. Since (38) is true for $k = 0$, it remains to be shown that the RHS of (39) converges to zero as $n \to \infty$ for every integer $0 < k < N_i$ if and only if for every integer $p > 0$, at least one of the conditions of part (ii) are true. The rest of the proof of part (ii) is very similar to that of part (iv). The reader is referred to the proof of part (i) of Theorem 1 in [9] for further details. The proof of part (i) is slightly different with s_{n-p} playing the role of d_{n-p}. By assumption, d_n are independent of s_{n-p} for all integers p. Therefore, the $r = p$ terms need not be treated separately in (39). Consequently, the conditions of part (i) do not require that $s = s(p) \neq p$. ■

Proof of Lemma 2: Note the following property about the impulse responses, $w_{X,r}$ and $t_{Y,r}$

$$t_{Y+1,r} - t_{Y+1,r-1} = t_{Y,r-1} \quad \text{and} \quad w_{X+1,r} - w_{X+1,r-1} = w_{X,r-1} \tag{40}$$

The proof is by contradiction. Suppose that $\langle k_1 w_{X,r} + (W/T)k_2 t_{Y,r+p}\rangle_W$ does not converge to 0 as $r \to \infty$ for every integer p and every integer pair $(k_1, k_2) \neq (0,0)$ such that

$0 \leqslant k_1 < W$ and $0 \leqslant k_2 < T$. Then some integers p, k_1, k_2 exist such that $k_1 + k_2 > 0, k_1 < W$, and $k_2 < T$, and

$$k_1 w_{X,r} + (W/T) k_2 t_{Y,r+p} = m_r W \qquad \forall r \geqslant r_0. \tag{41}$$

Therefore, the following is true for every pair of integers $s, s+1$ such that $s > \max(r_0, c, d-c)$

$$\begin{aligned} k_1 w_{2,s+1} + (W/T) k_2 t_{Y,s+p+1} &= m_{s+1} W, \qquad m_{s+1} \in \mathbb{Z} \\ k_1 w_{2,s} + (W/T) k_2 t_{Y,s+p} &= m_s W, \qquad m_s \in \mathbb{Z}. \end{aligned} \tag{42}$$

Subtracting the second of the above pair of equations from the first, and substituting (40) yields

$$k_1 w_{X-1,s} + (W/T) k_2 t_{Y-1,s+p} = l_s W \qquad \forall s > \max(r_0, c, d-c) \tag{43}$$

where l_s is an integer. Note that (41) and (43) together imply a set of equations that can be recursively applied till either X or Y becomes zero. Since $X < Y$, proceeding recursively gives

$$k_1 + (W/T) k_2 t_{Y-X,s+p} = n_s W \qquad \forall s \geqslant \max(r_0, c, d-c) \tag{44}$$

where the fact that $w_{0,r} = u_{r-c} = 1$ for large r has been used. The difference of two equations derived from (44) for every pair of consecutive integers is

$$(W/T) k_2 t_{Y-X-1,s+p} = q_s W, \qquad q_s \in \mathbb{Z} \qquad \forall s \geqslant \max(r_0, c, d-c). \tag{45}$$

Applying the resulting recursive equations till the first index variable on t reaches 0 results in

$$(W/T) k_2 t_{0,s+p} = (W/T) k_2 = \tilde{q} W, \qquad \tilde{q} \in \mathbb{Z}. \tag{46}$$

which can not true since $0 \leqslant k_2 < W$ and W/T is an integer. The contradiction proves the result. ■

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