A Fundamental Limitation of DC-Free Quantization Noise With Respect To Nonlinearity-Induced Spurious Tones

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***Abstract*—Fractional-N phase-locked loops (PLLs) are widely used to synthesize local oscillator signals for modulation and demodulation in communication systems. Such PLLs generate and subsequently lowpass filter DC-free quantization noise as part of their normal operation. Unfortunately, the quantization noise and its running sum inevitably are subjected to nonlinear distortion from analog circuit imperfections which causes spurious tones in the PLL output signal that can degrade communication system performance. This paper presents the first general mathematical analysis of this phenomenon. It proves that if the running sum of the quantization noise, $t[n]$, satisfies $t_{low} < t[n] \leq t_{high}$ for all n, where t_{low} and t_{high} are integers, then subjecting $t[n]$ to kth-order distortion for at least one $k \in \{1, 2, 3, \ldots, t_{high} - t_{low}\}$ will result in spurious tones for most fractional-N PLL output frequencies regardless of how the quantization is performed. It also shows that quantizers exist which are optimal in the sense that subjecting the running sum of their quantization noise to kth-order distortion for any $k \in \{1, 2, 3, \ldots, t_{high} - t_{low} - 1\}$ does not result in any spurious tones. In a typical fractional-N PLL, the larger the range of $t[n]$ the greater the power of the PLL's phase noise, so these results imply a fundamental tradeoff between phase noise power and spurious tones in PLLs.**

***Index Terms*—DC-free quantization noise, noise-shaping quantizers, phase-locked loops, spurious tones.**

I. Introduction

FRACTIONAL-N phase-locked loops (PLLs) are widely used to synthesize local oscillator signals for modulation and demodulation in communication systems, as they can provide fine frequency tuning resolution with relatively low power consumption and integrated circuit area [1], [2]. Ideally, a fractional-N PLL's output signal is perfectly periodic, so its phase increases linearly with time. Unfortunately, non-ideal circuit behavior causes the actual phase of the output signal to deviate from its ideal phase, where the deviation is referred to as *phase noise*. The phase noise inevitably consists of both periodic components called spurious tones and random components. Spurious tones are particularly harmful to the performance of typical communication systems, so most communication standards directly or indirectly stipulate stringent limits on the maximum power of the spurious tones in addition to specifying the maximum tolerable power of the overall phase noise in relevant frequency bands [3].

Fractional-N PLLs generally contain noise-shaping coarse quantizers, most commonly implemented as digital delta-sigma ($\Delta\Sigma$) modulators, which have recently been shown to be a significant, albeit indirect, source of phase noise spurious tones [4]–[10]. The output frequency of a fractional-N PLL is controlled by a digital codeword that represents a rational number, α, between 0 and 1. The coarse quantizer operates on α and generates a digital sequence that can be viewed as the sum of α and DC-free *quantization noise* [11]–[13].[1] The quantization noise is converted into analog form, integrated, and lowpass filtered within the PLL, and the resulting waveform directly adds to the PLL phase noise [1]. Unfortunately, the quantization noise and its running sum are subjected to nonlinear distortion from inevitable analog circuit imperfections, and this can induce spurious tones even when the quantization noise itself is free of spurious tones.

This problem is mitigated in the fractional-N PLL presented in [7] wherein the *successive requantizer* proposed in [6] is used in place of a $\Delta\Sigma$ modulator. The successive requantizer offers the advantage that its quantization noise and the running sum of its quantization noise remain free of spurious tones even when subjected to the type of nonlinear distortion commonly imposed by non-ideal circuit behavior in PLLs. This enables the PLL presented in [7] to achieve state-of-the-art spurious tone performance, but a price is paid for this benefit. In return for the enhanced immunity to nonlinearity-induced spurious tones, the power of the quantization noise introduced by the successive requantizer is significantly higher than that of a comparable $\Delta\Sigma$ modulator. The PLL presented in [7] employs a technique known as phase noise cancellation to overcome this problem at the expense of additional power consumption and circuit area.

No previous publications have addressed the question of whether the tradeoff between immunity to nonlinearity-induced spurious tones and increased quantization noise power observed in the successive requantizer is inevitable. This is an important question because if the tradeoff were just an idiosyncrasy of the successive requantizer, it might be possible to design an

[1] A sequence whose running sum is bounded for all time is said to be DC-free.

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improved coarse quantizer with good immunity to spurious tones that is not subject to the tradeoff. This paper answers this question.

The results of the paper prove that spurious tones are inevitably generated for most values of α when the running sum of DC-free quantization noise from a quantizer operating on α is subjected to the type of nonlinear distortion typically imposed by fractional-N PLLs. Specifically, if the running sum of the quantization noise, $t[n]$, satisfies $t_{low} < t[n] \leq t_{high}$ for all n, where t_{low} and t_{high} are integers, then subjecting $t[n]$ to kth-order distortion for at least one k in the set $\{1, 2, 3, \ldots, t_{high} - t_{low}\}$ will result in spurious tones for most values of α regardless of how the quantization is performed. The paper also shows that quantizers exist which are optimal in the sense that subjecting the running sum of their quantization noise to kth-order distortion for any value of k in the set $\{1, 2, 3, \ldots, t_{high} - t_{low} - 1\}$ does not result in any spurious tones. Therefore, the results imply a fundamental tradeoff between phase noise power and spurious tone suppression in a PLL.

The remainder of the paper consists of three main sections. Section II describes the details of the spurious tone problem in fractional-N PLLs, Section III presents and proves the theoretical results outlined above, and Section IV presents a method of quantization that is optimal in the sense described above.

II. Spurious Tones in Fractional-N PLLs

A. A Tone Definition Based on the Periodogram

Consider a discrete-time complex-coefficient image-rejection bandpass filter with a positive-frequency passband centered at any non-zero frequency ω_p and an adjustable equivalent noise bandwidth, $\Delta\omega_p$, wherein the passband's peak power gain times $\Delta\omega_p$ is unity. A sequence applied to the filter is said to contain a tone at ω_p if the squared magnitude of the output of the filter grows without bound as $\Delta\omega_p$ is reduced to zero. This description of a tone is consistent with the way that tones are measured in the laboratory using a spectrum analyzer [14].

An example of such a bandpass filter has a length-L impulse response given by

$$h_L[n] = \begin{cases} \frac{1}{\sqrt{L}} e^{-j\omega_p(L-1-n)} & \text{if } 0 \leq n \leq L-1, \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

where $\Delta\omega_p$ goes to zero as L goes to infinity. If the filter is applied to a sequence, $x[n]$, the squared magnitude of the filter output at time index $n = L - 1$ can be written as

$$I_{x,L}(\omega_p) = \frac{1}{L} \left| \sum_{k=0}^{L-1} x[k] e^{-j\omega_p k} \right|^2. \tag{2}$$

The expression given by (2) for any positive integer L and any $0 \leq |\omega_p| \leq \pi$ is known as the periodogram [15]. Therefore, the periodogram performs a function analogous to that of a laboratory spectrum analyzer, where increasing L in the periodogram is akin to decreasing the resolution bandwidth of the spectrum analyzer.

Accordingly, a mathematical definition of a tone that reflects the way that tones are measured in the laboratory is as follows.

Definition: Given any $\omega_p \neq 0$, $x[n]$ contains a tone at ω_p if $I_{x,L}(\omega_p)$ is unbounded as $L \to \infty$.

The definition implies that a sequence $x[n]$ is free of tones if and only if $I_{x,L}(\omega)$ is bounded in L for all $0 < |\omega| \leq \pi$.[2]

B. Spurious Tone Generation in Fractional-N PLLs

Ideally, a fractional-N PLL generates a periodic output signal $v_{out}(t)$ with frequency $f_{PLL} = f_{ref}(N + \alpha)$, where f_{ref} is the frequency of a reference oscillator, N is an integer, and $0 \leq \alpha < 1$. In practice, however, the output signal is more accurately modeled by

$$v_{out}(t) = g\left(2\pi f_{PLL} t + \theta_{PLL}(t)\right), \tag{3}$$

where g is a 2π-periodic function and $\theta_{PLL}(t)$ is the phase noise of the PLL [16].

As shown in Fig. 1, a typical fractional-N PLL consists of a phase detector, a lowpass loop filter, a voltage controlled oscillator (VCO), a frequency divider, and a noise-shaping coarse quantizer that introduces DC-free quantization noise. The phase detector drives the loop filter with a signal that represents the phase difference between the reference oscillator and frequency divider outputs. The instantaneous frequency of the VCO output signal deviates from its center frequency by an amount proportional to the output of the loop filter at each point in time. The frequency divider output is a two-level signal in which the nth and $(n+1)$th rising edges are separated by $N+y[n]$ cycles of the VCO output, where $y[n]$ for each n is an integer generated by the coarse quantizer. The PLL feedback loop adjusts the output frequency so as to zero the DC component of the phase detector output, causing the output frequency to settle to f_{ref} times the average of $N + y[n]$. If $y[n]$ could be set to α for all n, the PLL would have the desired output frequency. However, practical frequency dividers can only count integer numbers of VCO cycles, so $y[n]$ must be integer-valued. Therefore, the coarse quantizer ensures that $y[n]$ is integer-valued but averages to α in time. This results in the desired PLL output frequency, but the deviations of $y[n]$ from α contribute an extra component to the PLL's phase noise.

In general, $y[n]$ can be viewed as a representation of α quantized to be integer valued, and thus can be written as $y[n] = \alpha + s[n]$, where $s[n]$ is the quantization noise of $y[n]$. As explained in the introduction, it is desirable to engineer both $s[n]$ and its running sum $t[n]$, defined by

$$t[n] = \sum_{k=0}^{n} s[k], \tag{4}$$

to be free of spurious tones and also such that sequences resulting from nonlinearly distorting $s[n]$ and $t[n]$ are free of spurious tones. In practice, it is most critical for $t[n]$ to have these

[2]An alternative definition of a tone could be constructed based on traditional power spectral density (PSD) functions. However, the periodogram-based definition is preferred in this work for two reasons. First, the periodogram can be computed for any signal, whereas the PSD is only defined for a relatively small class of signals. Second, the phase noise performance of PLLs is usually quantified by time averages using laboratory equipment such as spectrum analyzers, not by ensemble averages. In this sense, the periodogram provides a meaningful representation of the power spectrum as used in practice.

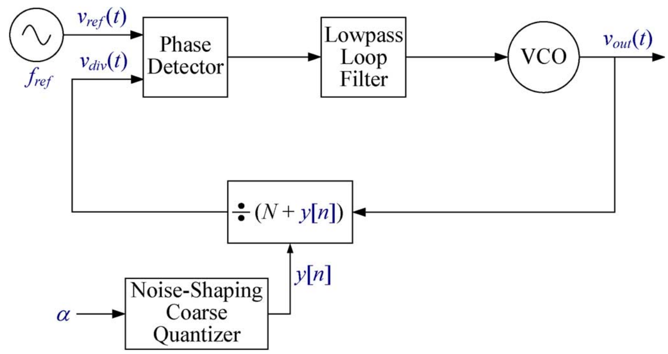


Fig. 1. Block diagram of a fractional-N PLL.

properties, because spurious tones generated by nonlinearly distorting $s[n]$ usually can be sufficiently mitigated by well-known frequency divider linearization techniques such as periodically resynchronizing each frequency divider output rising edge to the next rising edge of the VCO output signal [17].

As described in the introduction, it is usually highly undesirable for the phase noise of a PLL to contain tones, so any tones in a PLL's phase noise are usually referred to as *spurious tones*. Given that this paper describes a spurious tone generation mechanism in PLLs, all tones in the following will be denoted as spurious tones. Nevertheless, from a mathematical point of view there is no distinction between tones and spurious tones.

A sequence $x[n]$ is said to be *immune to spurious tones up to order* h if $x^p[n]$ is free of tones for all positive integers $p \leq h$. Based on simulation and experimental results, the nonlinearities to which $t[n]$ is subjected in a PLL tend to be well-modeled as truncated memoryless power series, i.e. functions of the form

$$f\left(t[n]\right) = a_0 + a_1 t[n] + a_2 t^2[n] + \ldots + a_k t^k[n] \tag{5}$$

for some positive integer k [6], [7], [18]. Thus, mitigating spurious tone generation in a PLL can be achieved by ensuring that $t[n]$ is immune to spurious tones up to a certain order. As explained in the introduction, $s[n]$ is required to be DC-free, which means that $t[n]$ is bounded, so

$$t_{low} < t[n] \leq t_{high} \tag{6}$$

for all n, where t_{low} and t_{high} are integers. Larger values of $t_{high} - t_{low}$ offer more flexibility in the design of the coarse quantizer, which can be exploited to increase the order of the spurious tone immunity of $t[n]$. The results presented in Section III show that the maximum attainable order of spurious tone immunity $t[n]$ can achieve is bounded by $(t_{high} - t_{low} - 1)$ regardless of how the quantization is performed.

III. Theory of Spurious Tones in DC-Free Quantization Noise

The following theorem shows that it is not possible to quantize most values of α such that the quantization noise is DC-free and its running sum is immune to spurious tones up to order $t_{high} - t_{low}$. The result is general in that it holds regardless of how the quantization is performed.

Theorem: Let α be a constant that satisfies $0 < \alpha < 1$, let $s[n]$ be a sequence such that

$$y[n] = \alpha + s[n] \tag{7}$$

is integer-valued for all n, and let

$$t[n] = \sum_{k=0}^{n} s[k]. \tag{8}$$

If

$$t_{low} < t[n] \leq t_{high} \tag{9}$$

for all n, where t_{low} and t_{high} are integers, and

$$\alpha = \frac{P}{Q}, \tag{10}$$

where P and Q are relatively prime integers with $Q > t_{high} - t_{low}$, then

$$I_{t^p,L}(2\pi f) = \frac{1}{L}\left|\sum_{n=0}^{L-1} t^p[n] e^{-j2\pi f n}\right|^2 \tag{11}$$

is unbounded in L for at least one $p \in \{1, 2, \ldots, t_{high} - t_{low}\}$ and at least one $f \in \{\alpha, 2\alpha, \ldots, (Q-1)\alpha\}$.

A practical implication of the theorem is that trying to develop a coarse quantizer applicable to fractional-N PLLs that eliminates the spurious tone generation mechanism described in Section II-B is futile. The coarse quantizer in any fractional-N PLL consists entirely of digital logic and its variables are represented by finite-width data buses, so all variables associated with the coarse quantizer, including α, are rational numbers. In particular, this implies that α satisfies (10). Furthermore, the coarse quantizer in a fractional-N PLL is required to have DC-free quantization noise. Thus, any coarse quantizer applicable to a fractional-N PLL must satisfy the theorem's hypothesis. The theorem places no other restrictions on the quantizer; the quantization noise can be deterministic or probabilistic and the theorem does not make any assumptions whatsoever about the quantizer's structure.

Another practical implication of the theorem is that the order of immunity to nonlinearity-induced spurious tones of $t[n]$ from the coarse quantizer in a fractional-N PLL can only be increased at the expense of increasing the range of values spanned by $t[n]$. The sequence $t[n]$ can be viewed as a lowpass filtered version of the quantization noise, so increasing its range tends to increase the power of the quantization noise at low frequencies where the PLL's loop filter provides little or no attenuation. The portion of $t[n]$ within a fractional-N PLL's bandwidth is an additive component of the PLL's phase noise, so all other things being the same, increasing the range of $t[n]$ increases power of the PLL's phase noise [1]. Furthermore, most integrated circuit based fractional-N PLLs use a phase-frequency detector and charge pump to implement the phase detector in Fig. 1, so the larger the magnitude of $t[n]$ at any time index n, the longer the current sources in the charge pump are turned on during the nth reference period. Increasing the on-time of the current sources causes more of the current source noise to be converted to phase noise, so all other things being the same, increasing the range of $t[n]$ also increases the power of the phase noise component contributed by the charge pump.

Proof of the Theorem: Equations (7) and (8) imply that

$$t[n] = \sum_{k=0}^{n} y[k] - (n+1)\alpha, \tag{12}$$

which can be written as

$$t[n] = \sum_{k=0}^{n} y[k] - \lfloor (n+1)\alpha \rfloor - \langle (n+1)\alpha \rangle, \tag{13}$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x and $\langle x \rangle$ denotes the fractional part of x, i.e. $\langle x \rangle = x - \lfloor x \rfloor$. Let

$$r[n] = \sum_{k=0}^{n} y[k] - \lfloor (n+1)\alpha \rfloor \tag{14}$$

with which (13) can be written as

$$t[n] = r[n] - \langle (n+1)\alpha \rangle. \tag{15}$$

By definition, $r[n]$ is an integer-valued sequence.[3] Furthermore,

$$r[n] \in \{t_{low}+1, t_{low}+2, \ldots, t_{high}\} \tag{16}$$

for all n, because $t[n]$ is bounded according to (9) and the last term in (15) is non-negative and less than 1.

Let

$$I_{t^p,L}(\omega) = \frac{1}{L}\left|\sum_{n=0}^{L-1} t^p[n] e^{-j\omega n}\right|^2. \tag{17}$$

Substituting (15) into (17) results in

$$I_{t^p,L}(\omega) = \frac{1}{L}\left|\sum_{n=0}^{L-1} \left(r[n] - \langle (n+1)\alpha \rangle\right)^p e^{-j\omega n}\right|^2. \tag{18}$$

Let $L = RQ$, where R is any positive integer, and $\omega = 2\pi i/Q$, where $i \in \{1, 2, \ldots, Q-1\}$. Then (18) can be written as

$$I_{t^p,RQ}\left(\frac{2\pi i}{Q}\right) = \frac{1}{RQ}\left|\sum_{k=0}^{R-1}\sum_{n=0}^{Q-1}\left(r[kQ+n] - \langle (kQ+n+1)\alpha \rangle\right)^p e^{-j\frac{2\pi i}{Q}(kQ+n)}\right|^2. \tag{19}$$

Given that $\alpha = P/Q$, where P and Q are relatively prime integers (so they have no common integer factors other than 1), the smallest value of n greater than zero for which $n\alpha$ is integer-valued is Q. Therefore, $\langle (n+1)\alpha \rangle$ is a periodic sequence with period Q, so

$$\langle (kQ+n+1)\alpha \rangle = \langle (n+1)\alpha \rangle \tag{20}$$

for each integer k. Substituting (20) into (19), interchanging the summations, and rearranging factors gives

$$I_{t^p,RQ}\left(\frac{2\pi i}{Q}\right) = \frac{R}{Q}\left|\sum_{n=0}^{Q-1}\left[\sum_{k=0}^{R-1}\frac{1}{R}\left(r[kQ+n] - \langle (n+1)\alpha \rangle\right)^p\right] e^{-j\frac{2\pi i}{Q}n}\right|^2. \tag{21}$$

Given that $r[kQ+n]$ is integer-valued and bounded according to (16), this can be rewritten as

$$I_{t^p,RQ}\left(\frac{2\pi i}{Q}\right) = \frac{R}{Q}\left|\sum_{n=0}^{Q-1}\left[\sum_{m=t_{low}+1}^{t_{high}} P_R[m,n]\left(m - \langle (n+1)\alpha \rangle\right)^p\right] e^{-j\frac{2\pi i}{Q}n}\right|^2, \tag{22}$$

where

$$P_R[m,n] = \frac{1}{R}\sum_{k=0}^{R-1}\gamma[k,m,n], \tag{23}$$

and

$$\gamma[k,m,n] = \begin{cases} 1, & \text{if } r[kQ+n] = m, \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

The summation in (23) counts the number of times that $r[kQ+n] = m$ over the R consecutive values of k from 0 to $R-1$. It follows that $P_R[m,n]$ has the same properties as a probability distribution in m for each n and each R, i.e.,

$$0 \le P_R[m,n] \le 1 \tag{25}$$

and

$$\sum_{m=t_{low}+1}^{t_{high}} P_R[m,n] = 1. \tag{26}$$

Equation (22) can be rewritten as

$$I_{t^p,RQ}\left(\frac{2\pi i}{Q}\right) = \frac{R}{Q}\left|\sum_{n=0}^{Q-1}\beta_R^{(p)}[n] e^{-j\frac{2\pi i}{Q}n}\right|^2, \tag{27}$$

where

$$\beta_R^{(p)}[n] = \sum_{m=t_{low}+1}^{t_{high}} P_R[m,n]\left(m - \langle (n+1)\alpha \rangle\right)^p. \tag{28}$$

Thus, the right side of (27) is R/Q times the squared magnitude of the discrete Fourier transform (DFT) of $\beta_R^{(p)}[n]$. A necessary condition for the DFT of $\beta_R^{(p)}[n]$, i.e.,

$$\sum_{n=0}^{Q-1}\beta_R^{(p)}[n] e^{-j\frac{2\pi i}{Q}n}, \tag{29}$$

[3] It follows from this and (15) that the fractional part of $t[n]$ is periodic, so it consists entirely of spurious tones. The fractional part operator is a memoryless nonlinearity, so this demonstrates that it is not possible for $t[n]$ to be immune to spurious tones for all memoryless nonlinearities.

to converge to 0 for every $i = 1, 2, \ldots, Q-1$ as R goes to infinity, and, therefore, for R/Q times the DFT of $\beta_R^{(p)}[n]$ to be bounded in R for every $i = 1, 2, \ldots, Q-1$, is

$$\beta_R^{(p)}[n] \rightarrow b_p \qquad \text{as} \qquad R \rightarrow \infty, \tag{30}$$

where b_p does not depend on n. Given that $L = RQ$, it follows that (30) is also a necessary condition for (11) to remain bounded in L.

Suppose the theorem is false. Then the above implies that there must exist Q probability distributions in m, $P[m,n]$ for $n = 0, 1, 2, \ldots, Q-1$, each of which must satisfy

$$\sum_{m=t_{low}+1}^{t_{high}} P[m,n] \left(m - \langle (n+1)\alpha \rangle\right)^p = b_p \tag{31}$$

for $p = 1, 2, \ldots, t_{diff}$, where

$$t_{diff} = t_{high} - t_{low}. \tag{32}$$

Additionally, given that $P[m,n]$ for $n = 0, 1, 2, \ldots, Q-1$ are probability distributions, (31) must hold for $p = 0$ and $b_0 = 1$. Thus, (31) represents $Q(t_{diff}+1)$ equations that must be satisfied by $t_{diff}Q$ probability values and t_{diff} values of b_p. This can be viewed a linear system of $Q(t_{diff}+1)$ equations with $t_{diff}Q + t_{diff}$ unknowns. With $Q > t_{diff}$, the system has more equations than unknowns, so if the theorem is false the equations must be linearly dependent.

The equations represented by (31) for each $n \in \{0, 1, 2, \ldots, Q-1\}$ and all $p \in \{0, 1, \ldots, t_{diff}-1\}$ can be written as

$$\mathbf{M}(x)\mathbf{p}(x) = \mathbf{b} \tag{33}$$

with values of x given by

$$x = \langle (n+1)\alpha \rangle, \tag{34}$$

where $\mathbf{M}(x)$ is given by (35), $\mathbf{b}$ is given by (36), and

$$\mathbf{p}(\langle (n+1)\alpha \rangle) = \left(P[t_{low}+1,n] \; P[t_{low}+2,n] \; \ldots \; P[t_{high},n]\right)^T. \tag{37}$$

Furthermore, the equations represented by (31) for each $n \in \{0, 1, 2, \ldots, Q-1\}$ and $p = t_{diff}$ can be written as

$$\mathbf{m}_{t_{diff}}(x)\mathbf{p}(x) = b_{t_{diff}}, \tag{38}$$

with x given by (34) and

$$\mathbf{m}_{t_{diff}}(x) = \left((t_{low}+1-x)^{t_{diff}} \; (t_{low}+2-x)^{t_{diff}} \; \ldots \; (t_{high}-x)^{t_{diff}}\right). \tag{39}$$

It follows from the lemma presented in the Appendix that $\mathbf{m}_{t_{diff}}(x)$ can be expressed in terms of $\mathbf{M}(x)$ as

$$\mathbf{m}_{t_{diff}}(x) = \mathbf{r}(x)\mathbf{M}(x), \tag{40}$$

where the kth element of $\mathbf{r}(x)$ is given by

$$(-1)^{t_{diff}-k} \sum_{1 \le i_1 < i_2 < \ldots < i_{t_{diff}-k+1} \le t_{diff}} y_{i_1}(x) y_{i_2}(x) \cdots y_{i_{t_{diff}-k+1}}(x) \tag{41}$$

with

$$y_{i_q}(x) = (t_{low} + i_q - x). \tag{42}$$

Therefore, (38) and (40) imply

$$b_{t_{diff}} = \mathbf{r}(x)\mathbf{M}(x)\mathbf{p}(x). \tag{43}$$

Substituting (33) into this result yields

$$b_{t_{diff}} = \mathbf{r}(x)\mathbf{b}. \tag{44}$$

If the theorem is false, (44) must hold for all values of x in the set

$$\{\langle \alpha \rangle, \langle 2\alpha \rangle, \ldots, \langle Q\alpha \rangle\}, \tag{45}$$

with $Q > t_{diff}$. The set contains Q distinct values of x because P and Q are relatively prime integers, so (44) must hold for more than t_{diff} distinct values of x if the theorem is false. It follows from (41) and (42) that the first element of $\mathbf{r}(x)$ is a polynomial in x of degree t_{diff}, and each of the other elements of $\mathbf{r}(x)$ is a polynomial in x of degree less than t_{diff}. Given that the first element of $\mathbf{b}$ is non-zero, this implies that

$$\mathbf{r}(x)\mathbf{b} - b_{t_{diff}} \tag{46}$$

is a polynomial of degree t_{diff}. Therefore, (46) has t_{diff} roots, so there can be at most t_{diff} distinct values of x that satisfy (44). This contradicts the supposition that the theorem is false. ∎

The theorem presented above implies that it is not possible to quantize most values of α such that the quantization noise is

$$\mathbf{M}(x) = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ (t_{low}+1-x) & (t_{low}+2-x) & \ldots & (t_{high}-x) \\ (t_{low}+1-x)^2 & (t_{low}+2-x)^2 & \ldots & (t_{high}-x)^2 \\ \ldots & \ldots & \ldots & \ldots \\ (t_{low}+1-x)^{t_{diff}-1} & (t_{low}+2-x)^{t_{diff}-1} & \ldots & (t_{high}-x)^{t_{diff}-1} \end{pmatrix}, \tag{35}$$

$$\mathbf{b} = \left(1 \quad b_1 \quad b_2 \quad \ldots \quad b_{t_{diff}-1}\right)^T, \tag{36}$$

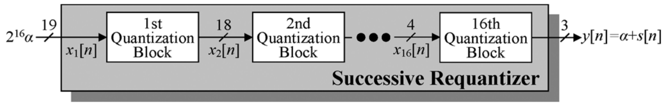


Fig. 2. High-level diagram of an example successive requantizer.

DC-free and its running sum is immune to spurious tones up to order $t_{high} - t_{low}$. As explained below, this bound on performance is tight in the sense that quantizers exist with the property that the running sum of their quantization noise is immune to spurious tones up to order $t_{high} - t_{low} - 1$. The theorem implies that a quantizer with this property is optimal with respect to spurious tone immunity in the sense that the running sum of its quantization noise has the highest possible order of immunity to spurious tones.

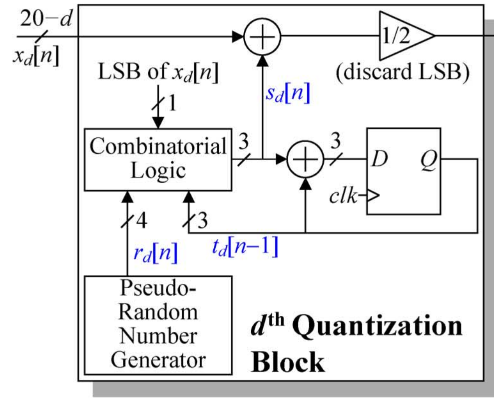
The successive requantizer provides an existence proof that quantizers exist which are optimal with respect to spurious tone immunity. As an example, the successive requantizer proposed in [6] and demonstrated in the fractional-N PLL integrated circuit presented in [7] is shown in Figs. 2 and 3. For this particular successive requantizer, α can be any multiple of 2^{-16} that is non-negative and less than 1. As shown in Fig. 2, the successive requantizer multiples α by 2^{16} and processes the integer-valued result via a cascade of 16 *quantization blocks*, each of which simultaneously quantizes by one bit and halves each sample of its input sequence. The implementation details of each quantization block are shown in Fig. 3. As proven in [6], $y[n]$ is an integer-valued quantized version of α, its quantization noise is DC-free with $t_{high} - t_{low} = 4$, and the running sum of its quantization noise is immune to spurious tones up to order 3.

The results in [6] are extended in [19] to show that for each positive integer $N_t \geq 2$ there exist multiple successive requantizers that have $t_{high} - t_{low} = 2N_t$ and for which the running sum of their quantization noise is immune to spurious tones up to order $2N_t - 1$. Therefore, each of these successive requantizers is an optimal quantizer with respect to spurious tone immunity in the sense that the running sum of its quantization noise has the highest possible order of immunity to spurious tones.

While the theorem quantifies the relationship between the value of Q and the possible frequencies of the nonlinearity-induced spurious tones, it does not quantify the power of the nonlinearity-induced spurious tones. This is because the theorem is applicable to any quantizer with DC-free quantization noise, whereas the effect of varying Q on quantizer performance for a particular quantizer depends on the quantizer's design. For example, in most delta-sigma modulators with DC-free quantization noise the nonlinearity-induced spurious tone powers are strongly dependent on Q, whereas for the successive requantizer described above computer simulations suggest that they are nearly independent of Q. Thus, the effect of varying Q on quantizer performance must be evaluated in a quantizer specific fashion.

IV. Alternate Method of Optimal Quantization

The successive requantizer is not the only type of quantizer that is optimal with respect to spurious tone immunity. An alternate method of quantization that is optimal with respect to spurious tone immunity is presented in this section. Unlike the



Combinatorial Logic Truth Table:

LSB of $x_d[n]$ = 0		
$t_d[n-1]$	$r_d[n]$	$s_d[n]$
2	≥ 0 and ≤ 3	0
2	≤ -1 or ≥ 4	−2
1	≤ -1 or ≥ 6	0
1	≥ 0 and ≤ 5	−2
0	0 or 1	2
0	≤ -1 or ≥ 4	0
0	2 or 3	−2
−1	≤ -1 or ≥ 6	0
−1	≥ 0 and ≤ 5	2
−2	≥ 0 and ≤ 3	0
−2	≤ -1 or ≥ 4	2

LSB of $x_d[n]$ = 1		
$t_d[n-1]$	$r_d[n]$	$s_d[n]$
2	≤ -1 or ≥ 4	−1
2	≥ 0 and ≤ 3	−3
1	≥ 1 and ≤ 3	1
1	≤ -1 or ≥ 4	−1
1	0	−3
0	≥ 0	1
0	≤ -1	−1
−1	≥ 1 and ≤ 3	−1
−1	≤ -1 or ≥ 4	1
−1	0	3
−2	≤ -1 or ≥ 4	1
−2	≥ 0 and ≤ 3	3

Fig. 3. Details of each quantization block within the example successive requantizer.

successive requantizer, the idea upon which it is based follows directly from the proof of the theorem presented in Section III, so it gives some insight into the connection between the quantization process and the theorem.

Suppose that a quantized sequence with mean $\alpha = P/Q$, where P and Q are relatively prime integers, is to be generated, and that the running sum of the quantization noise is required to satisfy $-N_t < t[n] \leq N_t$ over all n for some positive integer N_t. Thus, $t_{low} = -N_t$ and $t_{high} = N_t$. By the analysis presented in the proof of the theorem up to (31), a necessary condition for $t[n]$ to be immune to spurious tones up to order $t_{high} - t_{low} - 1 = 2N_t - 1$ is that there exist Q probability distributions in m, $P[m, u]$, where $m \in \{-N_t + 1, -N_t + 2, \ldots, N_t\}$ and $u \in \{0, 1, 2, \ldots, Q-1\}$, which satisfy (31) for $p = 1, 2, \ldots, 2N_t - 1$. It follows from (31) that these probability distributions must satisfy

$$\sum_{m=-N_t+1}^{N_t} P[m,i]\,(m - \langle (i+1)\alpha\rangle)^p = \sum_{m=-N_t+1}^{N_t} P[m,i+1]\,(m - \langle (i+2)\alpha\rangle)^p, \quad (47)$$

for all $i \in \{0, 1, 2, \ldots, Q-2\}$, and $p \in \{1, 2, 3, \ldots, 2N_t - 1\}$. To be probability distributions, they must also be non-negative and satisfy

$$\sum_{m=-N_t+1}^{N_t} P[m,u] = 1, \quad (48)$$

for all $u \in \{0, 1, 2, \ldots, Q-1\}$. Any set of $P[m, n]$ that satisfy (47) and (48), can be used to generate $r[n]$ such that

$$r[n] \in \{-N_t + 1, -N_t + 2, \ldots, N_t\} \tag{49}$$

for all n and

$$\lim_{R \to \infty} \frac{1}{R} \sum_{k=0}^{R-1} \gamma[k, m, u] = P[m, u] \tag{50}$$

where

$$\gamma[k, m, u] = \begin{cases} 1, & \text{if } r[kQ + u] = m, \\ 0, & \text{otherwise.} \end{cases} \tag{51}$$

This can be done either probabilistically or deterministically. For each n, once $r[n]$ is known the running sum of the quantization noise, the quantization noise, and the quantizer output can be calculated using

$$t[n] = r[n] - \langle (n+1)\alpha \rangle, \tag{52}$$

$$s[n] = t[n] - t[n-1], \tag{53}$$

and

$$y[n] = \alpha + s[n], \tag{54}$$

respectively.

For instance, as done in the following examples, $r[n]$ can be generated as a sequence of independent random variables with probability distributions

$$\Pr(r[n] = m) = P[m, n \bmod Q] \tag{55}$$

for all $m \in \{-N_t + 1, -N_t + 2, \ldots, N_t\}$ and all integers n. It follows from (52) that $t^p[n]$ is a sequence of independent random variables, and from (47) that the mean of $t^p[n]$ is independent of n for $p \in \{1, 2, 3, \ldots, 2N_t - 1\}$. It follows that $t^p[n]$ is white noise and is therefore free of spurious tones for each $p \in \{1, 2, 3, \ldots, 2N_t - 1\}$.

There are many sets of non-negative $P[m, u]$ values that satisfy the system of equations specified by (47) and (48), because the system is under-constrained; it has $(Q-1)(2N_t - 1) + Q$ equations and $2N_t Q$ unknowns. Therefore, additional constraints can be imposed on the $P[m, u]$ values. For example, imposing additional constraints of the form

$$\sum_{m=-N_t+1}^{N_t} P[m, i] \left(m - \langle (i+1)\alpha \rangle\right)^{2N_t} = \sum_{m=-N_t+1}^{N_t} P[m, u] \left(m - \langle (u+1)\alpha \rangle\right)^{2N_t} \tag{56}$$

for as many $i, u \in \{0, 1, 2, \ldots, Q-1\}$ as possible has the effect of minimizing spurious tone power in $t^{2N_t}[n]$.

Two quantization noise running sum sequences, $t_1[n]$ and $t_2[n]$, based on the method described above are presented below and demonstrated by simulation to have optimal orders of spurious tone immunity. The magnitude bounds on $t_1[n]$ and $t_2[n]$ are $N_{t1} = 2$ and $N_{t2} = 3$, respectively, and the quantized sequences corresponding to $t_1[n]$ and $t_2[n]$ have means of $\alpha_1 = 1/5$ and $\alpha_2 = 1/7$, respectively. The $P[m, u]$ values found in both cases are presented in matrices $\mathbf{P}_1$ and $\mathbf{P}_2$, defined by

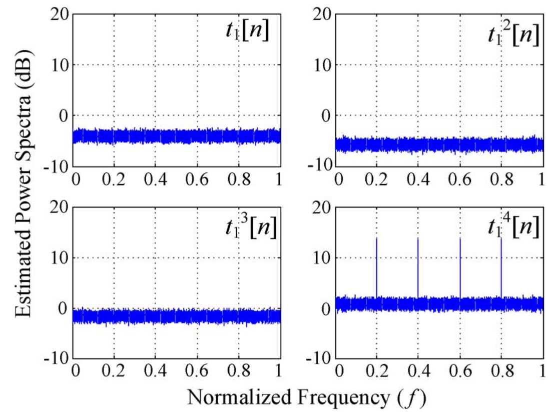


Fig. 4. Estimated power spectra of an optimal quantization noise running sum sequence bounded by 2 when raised to different powers.

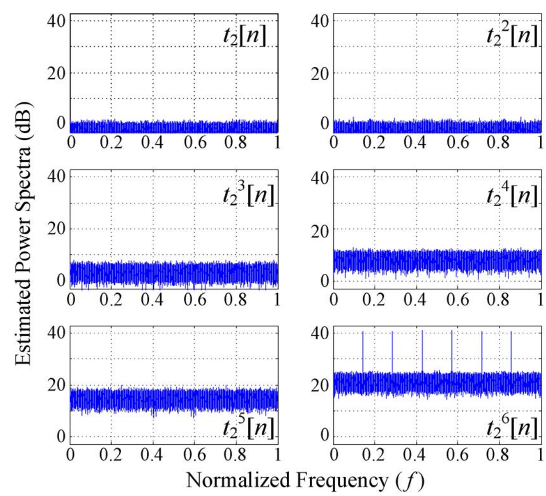


Fig. 5. Estimated power spectra of an optimal quantization noise running sum sequence bounded by 3 when raised to different powers.

$$\mathbf{P}_k(i, j) = P[-N_{tk} + i, j - 1] \tag{57}$$

for $i \in \{1, 2, \ldots, 2N_{tk}\}$, $j \in \{1, 2, \ldots, Q\}$, and $k = 1$ or 2:

$$\mathbf{P}_1 = \begin{pmatrix} \frac{14}{125} & \frac{7}{125} & \frac{3}{125} & \frac{1}{125} & \frac{1}{5} \\ \frac{73}{125} & \frac{64}{125} & \frac{51}{125} & \frac{37}{125} & \frac{3}{5} \\ \frac{37}{125} & \frac{51}{125} & \frac{64}{125} & \frac{73}{125} & \frac{1}{5} \\ \frac{1}{125} & \frac{3}{125} & \frac{7}{125} & \frac{14}{125} & 0 \end{pmatrix}, \tag{58}$$

$$\mathbf{P}_2 = \begin{pmatrix} \frac{132}{16807} & \frac{66}{16807} & \frac{30}{16807} & \frac{12}{16807} & \frac{4}{16807} & \frac{1}{16807} & \frac{5}{343} \\ \frac{137}{780} & \frac{2190}{16807} & \frac{273}{2945} & \frac{97}{1538} & \frac{687}{16807} & \frac{185}{7368} & \frac{78}{343} \\ \frac{236}{465} & \frac{880}{1821} & \frac{272}{611} & \frac{738}{1861} & \frac{860}{2519} & \frac{724}{2551} & \frac{177}{343} \\ \frac{724}{2551} & \frac{860}{2519} & \frac{738}{1861} & \frac{272}{611} & \frac{880}{1821} & \frac{236}{465} & \frac{78}{343} \\ \frac{185}{7368} & \frac{687}{16807} & \frac{97}{1538} & \frac{273}{2945} & \frac{2190}{16807} & \frac{137}{780} & \frac{5}{343} \\ \frac{1}{16807} & \frac{4}{16807} & \frac{12}{16807} & \frac{30}{16807} & \frac{66}{16807} & \frac{132}{16807} & 0 \end{pmatrix}. \tag{59}$$

Figs. 4 and 5 show the estimated power spectra of $t_1^p[n]$ for $p \in \{1,2,3,4\}$ and $t_2^q[n]$ for $q \in \{1,2,\ldots,6\}$. The figures demonstrate that spurious tones in $t_1^p[n]$ are present only when $p = 2N_{t1} = 4$ and that spurious tones in $t_2^q[n]$ are present only when $q = 2N_{t2} = 6$. This supports the assertion that both examples represent optimal quantization in terms of spurious tone immunity.

APPENDIX

The following lemma is used in the proof of the theorem in Section III.

Lemma: Given arbitrary $a_1, a_2, \ldots, a_n$, let $\mathbf{V}$ be the following $n \times n$ matrix:

$$\mathbf{V} = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ a_1 & a_2 & a_3 & \ldots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \ldots & a_n^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \ldots & a_n^{n-1} \end{pmatrix}. \quad (60)$$

Then, the row vector

$$\mathbf{v}_n = (a_1^n \; a_2^n \; a_3^n \; \ldots \; a_n^n) \quad (61)$$

can be expressed as

$$\mathbf{v}_n = \mathbf{r} \cdot \mathbf{V}, \quad (62)$$

where $\mathbf{r}$ is a row vector whose elements are given by

$$\begin{aligned} \mathbf{r}(1) &= (-1)^{n-1} \sum_{1 \le i_1 < i_2 < \ldots < i_n \le n} a_{i_1} a_{i_2} \cdots a_{i_n}, \\ \mathbf{r}(2) &= (-1)^{n-2} \sum_{1 \le i_1 < i_2 < \ldots < i_{n-1} \le n} a_{i_1} a_{i_2} \cdots a_{i_{n-1}}, \\ &\vdots \\ \mathbf{r}(n) &= \sum_{1 \le i_1 \le n} a_{i_1}. \end{aligned} \quad (63)$$

Proof: Consider the polynomial

$$P(x) = (x - a_1)(x - a_2) \cdots (x - a_n), \quad (64)$$

which can be expanded as

$$P(x) = x^n - x^{n-1} \sum_{1 \le i_1 \le n} a_{i_1} + x^{n-2} \sum_{1 \le i_1 < i_2 \le n} a_{i_1} a_{i_2} - \ldots + (-1)^n \sum_{1 \le i_1 < i_2 < \ldots < i_n \le n} a_{i_1} a_{i_2} \ldots a_{i_n}. \quad (65)$$

It follows from (64) that a_k is a root of $P(x)$ for any $k \in \{1, 2, \ldots, n\}$, i.e.

$$P(a_k) = 0. \quad (66)$$

Additionally, it is seen from (65) that $P(a_k)$ can be expressed as

$$P(a_k) = a_k^n - (\mathbf{r} \cdot \mathbf{V})(k), \quad (67)$$

where $(\mathbf{r} \cdot \mathbf{V})(k)$ is the kth element of the vector $\mathbf{r} \cdot \mathbf{V}$. Therefore, (66) and (67) yield

$$(\mathbf{r} \cdot \mathbf{V})(k) = a_k^n, \quad (68)$$

which proves the result. ■

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