A Class of Quantizers With DC-Free Quantization Noise and Optimal Immunity to Nonlinearity-Induced Spurious Tones

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***Abstract*—Fractional-N phase-locked loops (PLLs) typically use noise-shaping coarse quantizers to control their instantaneous output frequency. The resulting quantization noise and its running sum inevitably get distorted by non-ideal analog components within the PLL, which induces undesirable spurious tones in the PLL's output signal. A recently proposed quantizer, called a successive requantizer, has been shown to mitigate this problem. Its quantization noise and the running sum of its quantization noise can be subjected to up to fifth-order and third-order nonlinear distortion, respectively, without inducing spurious tones. This paper extends the previously published successive requantizer results to enable the design of successive requantizers whose quantization noise running sum sequences attain such immunity to nonlinearity-induced spurious tones up to arbitrarily high orders of distortion. The extended results are used to design example successive requantizers whose quantization noise and quantization noise running sum sequences have optimally reduced susceptibility to nonlinearity-induced spurious tones.**

***Index Terms*—DC-free quantization noise, noise-shaping quantizers, spurious tones.**

I. Introduction

FRACTIONAL-N phase locked loops (PLLs) are widely used to synthesize local oscillator signals in communication systems [1], [2]. They typically use noise-shaping coarse quantizers, most commonly implemented as digital delta-sigma ($\Delta\Sigma$) modulators, to quantize digital sequences that control their output frequency. Each quantized sequence can be viewed as the sum of the quantizer's input sequence plus *DC-free quantization noise* [3]–[5].[1] In practical PLLs, the quantization noise and its running sum inevitably are subjected to nonlinear distortion from analog circuit imperfections within the PLL. This has the undesirable effect of inducing spurious tones in the sequences, even when the undistorted sequences are free of spurious tones [6]–[12]. Spurious tones induced in this fashion are referred to as *nonlinearity-induced spurious tones*.

Most communication applications require the power of spurious tones to be very low, as they ultimately appear in the PLL's output signal and can be critically harmful to communication system performance. One way to minimize spurious tone power is to make the analog circuitry of the PLL very linear. Unfortunately, improving analog circuit linearity tends to increase power dissipation and integrated circuit area significantly. Alternatively, the coarse quantizer can be designed to ensure that the quantization noise and its running sum remain free of spurious tones even when subjected to the type of nonlinear distortion commonly imposed within the PLL.

A sequence $x[n]$ is said to be immune to spurious tones up to order h if $x^p[n]$, for $p = 1, 2, \ldots, h$, are free of spurious tones. A recently proposed quantizer, called a successive requantizer, was introduced in [8] and implemented as part of a phase-noise cancelling PLL in [9] to mitigate the power of nonlinearity-induced spurious tones. Its quantization noise and the running sum of its quantization noise are immune to spurious tones up to orders 5 and 3, respectively.

This paper extends the previously published successive requantizer results to design successive requantizers with higher immunity to nonlinearity-induced spurious tones. It proves that the order up to which the quantization noise running sum of a successive requantizer is immune to spurious tones can be arbitrarily increased at the expense of increasing the range of values spanned by the quantization noise running sum. In a PLL, increasing this range tends to increase the quantization noise power, and, therefore, the phase noise. Hence, a tradeoff exists between enhanced immunity to nonlinearity-induced spurious tones and increased phase noise power. The paper also presents successive requantizers that are optimal in the sense that their quantization noise and quantization noise running sum are immune to spurious tones up to the maximum possible orders for the range of values spanned by the quantization noise running sum.

II. Successive Requantizer Background

A. Spectral Properties of Interest

The periodogram of any sequence $x[n]$ is defined as

$$I_{x,L}(\omega) = \frac{1}{L}\left|\sum_{n=0}^{L-1} x[n]e^{-j\omega n}\right|^2 \tag{1}$$

for any positive integer L [13]. By definition, $x[n]$ contains a tone at $\omega_n \neq 0$ if $I_{x,L}(\omega)$ is unbounded at $\omega = \omega_n$ as $L \to \infty$ [8], [14]. In a PLL, the nonlinearities to which the quantization

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[1] A sequence whose running sum is bounded for all time is said to be DC-free.

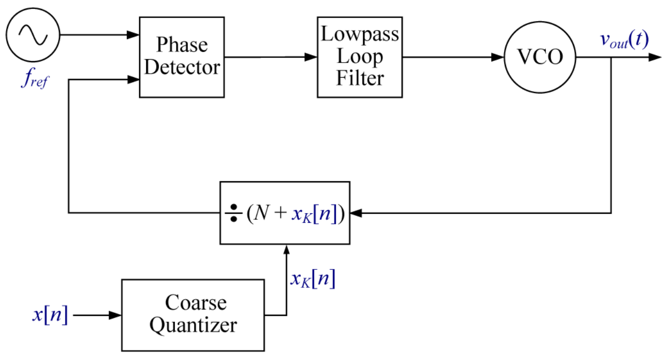

Fig. 1. Block diagram of a fractional-N PLL.

noise sequence $s[n]$ and the quantization noise running sum sequence

$$t[n] = \sum_{k=0}^{n} s[k] \tag{2}$$

are subjected tend to be well-modeled as truncated memoryless power series [9]. Therefore, this work focuses on the properties of $I_{s^q,L}(\omega)$ and $I_{t^p,L}(\omega)$ for integer values of q and p up to some maximum values.

B. Successive Requantizer Architecture

As shown in Fig. 1, a typical fractional-N PLL consists of a phase detector, a lowpass loop filter, a voltage controlled oscillator (VCO), a frequency divider, and a noise-shaping coarse quantizer that introduces DC-free quantization noise. Its purpose is to generate a periodic or frequency modulated output signal with an instantaneous frequency of $(N + x[n])f_{ref}$, where N is a positive integer, $x[n]$ is a sequence of fractional values, and f_{ref} is the frequency of a reference oscillator. In most applications $x[n]$ is constant, and in other applications it varies slowly. The PLL's feedback loop adjusts the output frequency to be f_{ref} times the average of the divider modulus $N + x_K[n]$. If $x_K[n]$ could be set to $x[n]$ for all n, the PLL would have the desired output frequency. However, practical frequency dividers can only count integer numbers of VCO cycles, so $x_K[n]$ must be integer-valued. Therefore, the coarse quantizer ensures that $x_K[n]$ is integer-valued but averages to $x[n]$ in time. This results in the desired PLL output frequency, although the deviations of $x_K[n]$ from $x[n]$ contribute an extra component to the PLL's phase noise. As explained in the introduction, the coarse quantizer can be implemented as the successive requantizer presented in [8].

The high-level architecture of the successive requantizer is shown in Figs. 2 and 3, wherein all node variables are integer-valued sequences in two's complement format. The successive requantizer consists of K serially-connected quantization blocks, each of which quantizes its input by 1 bit, so the successive requantizer quantizes its input by K bits. Its input,

$$x_0[n] = 2^K x[n], \tag{3}$$

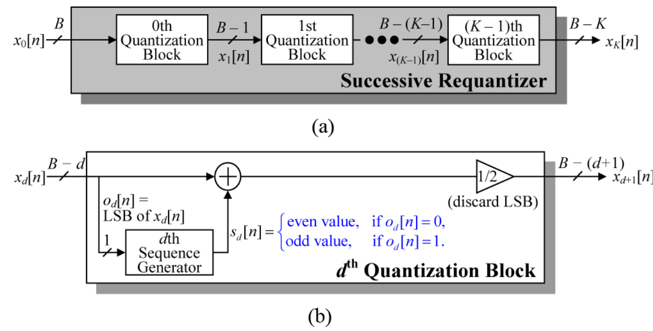

Fig. 2. High-level block diagram of a successive requantizer.

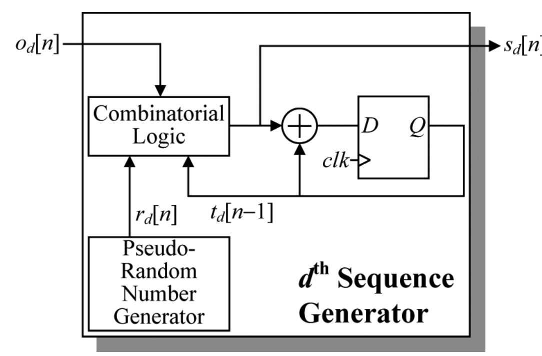

Fig. 3. Block diagram of a sequence generator.

is a B-bit sequence which satisfies $|x_0[n]| \le 2^{B-2}$ for all n. The dth quantization block's input, $x_d[n]$, and output, $x_{d+1}[n]$, are related through

$$x_{d+1}[n] = \frac{1}{2}\left(x_d[n] + s_d[n]\right), \tag{4}$$

where $s_d[n]$ is a sequence generated by the quantization block's *sequence generator*. The sequence generator (Fig. 3) generates $s_d[n]$ as a function of the *parity sequence*, $o_d[n]$, which at each time, n, is 1 if $s_d[n]$ is odd and 0 if $s_d[n]$ is even. It chooses $s_d[n]$ to have the same parity as $x_d[n]$ for each n so that $x_{d+1}[n]$ is an integer-valued sequence, and to have a sufficiently small magnitude that the two's complement representation of $x_{d+1}[n]$ requires one less bit than that of $x_d[n]$. Hence, the output of the successive requantizer is a two's complement integer-valued sequence given by

$$x_K[n] = 2^{-K}x_0[n] + s[n] = x[n] + s[n], \tag{5}$$

where

$$s[n] = \sum_{d=0}^{K-1} 2^{d-K} s_d[n] \tag{6}$$

is the quantization noise. The running sum of $s_d[n]$ is

$$t_d[n] = \sum_{k=0}^{n} s_d[k], \tag{7}$$

so (5) implies that the running sum of the quantization noise can be written as

$$t[n] = \sum_{d=0}^{K-1} 2^{d-K} t_d[n]. \tag{8}$$

The lowest integer bound on the magnitude of each $t_d[n]$ sequence is denoted as N_t, so $|t_d[n]| \le N_t$ for all d and n. Therefore, it follows from (7) that $|t[n]| < N_t$ for all n.

As shown in [8], if the sequence generator is designed such that the probability mass function (pmf) of $s_d[n]$ for each n depends only on $o_d[n]$ and $t_d[n-1]$, then $t_d[n]$ is a discrete-valued Markov random sequence conditioned on $o_d[n]$. Hence, for any parity sequence, $o_d[n]$, the evolution of $t_d[n]$ from times u to $u+m$ can be represented by an m-step $(2N_t+1)\times(2N_t+1)$ state transition matrix, $\mathbf{A}_{\{o_d[n]\}}[u,m]$, where the element on the ith row and jth column is

$$\left(\mathbf{A}_{\{o_d[n]\}}[u,m]\right)(i,j) = \Pr(t_d[u+m]=\mathbf{t}(j)|t_d[u]=\mathbf{t}(i), \quad o_d[n]; n=u+1, u+2, \ldots, u+m) \tag{9}$$

and

$$\mathbf{t} = (\, N_t \quad N_t-1 \quad \ldots \quad -N_t\,)^T. \tag{10}$$

It follows from the properties of state transition matrices that for $m>1$ $\mathbf{A}_{\{o_d[n]\}}[u,m]$ can be expanded as a product of one-step state transition matrices as

$$\mathbf{A}_{\{o_d[n]\}}[u,m] = \mathbf{A}_{\{o_d[n]\}}[u,1]\mathbf{A}_{\{o_d[n]\}}[u+1,1]\cdots\mathbf{A}_{\{o_d[n]\}}[u+m-1,1]. \tag{11}$$

As is also shown in [8], $\mathbf{A}_{\{o_d[n]\}}[v-1,1]$ at each time index v is equal to one of two one-step state transition matrices, denoted as $\mathbf{A_e}$ and $\mathbf{A_o}$: when $o_d[v]=0$, $\mathbf{A}_{\{o_d[n]\}}[v-1,1]=\mathbf{A_e}$, and when $o_d[v]=1$, $\mathbf{A}_{\{o_d[n]\}}[v-1,1]=\mathbf{A_o}$. It follows from (6) that

$$s_d[n] = t_d[n] - t_d[n-1], \tag{12}$$

so the $\mathbf{A_e}$ and $\mathbf{A_o}$ matrices describe the probabilistic behavior of each $s_d[n]$ sequence and determine the orders up to which $s[n]$ and $t[n]$ are immune to spurious tones. In any given successive requantizer they completely specify the required behavior of the combinatorial logic block and, conversely, can be deduced from the combinatorial logic block.

Each $t_d[n]$ sequence satisfies $|t_d[n]| \le N_t$ for all n, so it follows from (12) that each $s_d[n]$ sequence satisfies $|s_d[n]| \le 2N_t$ for all n. Therefore, (6) implies that $|s[n]| < 2N_t$ for all n, and that the output of the successive requantizer, given by (5), satisfies $|x_K[n]| \le 2N_t$ for all n. Since $x_K[n]$ is represented by a $(B-K)$-bit sequence,

$$B - K \ge \log_2(4N_t+1) \tag{13}$$

must hold.

Figs. 2 and 3 imply that the dth quantization block of the successive requantizer contains combinatorial logic that depends on the $\mathbf{A_e}$ and $\mathbf{A_o}$ matrices, a pseudo-random number generator, a $(B-d)$-bit adder, a $\lceil \log_2(4N_t+1)\rceil$-bit adder, and $\lceil \log_2(4N_t+1)\rceil$ flip flops, where $\lceil x \rceil$ denotes the smallest integer greater than x. With K blocks, where K is usually close to B in magnitude, the computational complexity of the successive requantizer is a logarithmic function of N_t and a quadratic function of B. As an example, the implementation of the successive requantizer in 0.18 μm 1P6M CMOS technology in [9], for which $N_t=3$ and $K=19$, and the related pseudo-random number generator, requires 1049 gates, 114 flip flops, and 232 1-bit adders, and occupies an area of 0.142 mm^2.

C. Example Successive Requantizers

If the combinatorial logic implements the truth table shown in Fig. 4(a), then $N_t=1$,

$$\mathbf{A_e} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } \mathbf{A_o} = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}. \tag{14}$$

It can be verified from the results presented in [15] that in this case $t[n]$ and $s[n]$ are free of spurious tones. However, the results presented in [14] and simulations support the conclusion that spurious tones are generated for some successive requantizer input sequences when $t[n]$ or $s[n]$ are subjected to second-order nonlinear distortion. Therefore, $t[n]$ and $s[n]$ are immune to spurious tones only up to order 1.

As proven in [9], if the combinatorial logic implements the truth table shown in Fig. 4(b), then $N_t=2$,

$$\mathbf{A_e} = \begin{pmatrix} 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 5/8 & 0 & 3/8 & 0 \\ 1/8 & 0 & 3/4 & 0 & 1/8 \\ 0 & 3/8 & 0 & 5/8 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \end{pmatrix}, \text{ and}$$

$$\mathbf{A_o} = \begin{pmatrix} 0 & 3/4 & 0 & 1/4 & 0 \\ 3/16 & 0 & 3/4 & 0 & 1/16 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 1/16 & 0 & 3/4 & 0 & 3/16 \\ 0 & 1/4 & 0 & 3/4 & 0 \end{pmatrix}. \tag{15}$$

In this case it follows from the results in [8] and [14] that $t[n]$ and $s[n]$ are immune to spurious tones up to orders 3 and 5, respectively.

D. Additional Successive Requantizer Properties

The $\mathbf{A_e}$ matrices in the examples presented above have the property that $\mathbf{A_e}(i,j)=0$ whenever $i+j$ is odd, and $\mathbf{A_o}(i,j)=0$ whenever $i+j$ is even; such matrices are referred to as even-entries matrices and odd-entries matrices, respectively. As is evident in the example $\mathbf{A_e}$ and $\mathbf{A_o}$ matrices presented above, the row vectors of both even-entries and odd-entries matrices alternate between two types of vectors: vectors whose odd-indexed elements are zero, referred to as even-entries vectors, and vectors whose even-indexed elements are zero, referred to as odd-entries vectors. For example, an even-entries vector $\mathbf{v_e}$ is such that $\mathbf{v_e}(i)=0$ whenever i is odd.

These properties of $\mathbf{A_e}$ and $\mathbf{A_o}$ hold in general as a result of (11). For any n at which $o_d[n]=0$, $s_d[n]$ is even, so (11) implies that the probability that $t_d[n]$ and $t_d[n-1]$ have different parities

Combinatorial Logic Truth Tables:

Case $N_t = 1$, $r_d[n]$ in {–1, 0}

$o_d[n] = 0$		
$t_d[n-1]$	$r_d[n]$	$s_d[n]$
1	−1 or 0	0
0	−1 or 0	0
−1	−1 or 0	0

$o_d[n] = 1$		
$t_d[n-1]$	$r_d[n]$	$s_d[n]$
1	−1 or 0	−1
0	−1	−1
0	0	1
−1	−1 or 0	1

(a)

Case $N_t = 2$, $r_d[n]$ in {–8, –7,..., 7}

$o_d[n] = 0$		
$t_d[n-1]$	$r_d[n]$	$s_d[n]$
2	≥ 0 and ≤ 3	0
2	≤ −1 or ≥ 4	−2
1	≤ −1 or ≥ 6	0
1	≥ 0 and ≤ 5	−2
0	0 or 1	2
0	≤ −1 or ≥ 4	0
0	2 or 3	−2
−1	≤ −1 or ≥ 6	0
−1	≥ 0 and ≤ 5	2
−2	≥ 0 and ≤ 3	0
−2	≤ −1 or ≥ 4	2

$o_d[n] = 1$		
$t_d[n-1]$	$r_d[n]$	$s_d[n]$
2	≤ −1 or ≥ 4	−1
2	≥ 0 and ≤ 3	−3
1	≥ 1 and ≤ 3	1
1	≤ −1 or ≥ 4	−1
1	0	−3
0	≥ 0	1
0	≤ −1	−1
−1	≥ 1 and ≤ 3	−1
−1	≤ −1 or ≥ 4	1
−1	0	3
−2	≤ −1 or ≥ 4	1
−2	≥ 0 and ≤ 3	3

(b)

Fig. 4. Example combinatorial logic truth tables of sequence generators corresponding to the one-step state transition matrices in (a) (12) and (b) (13). In both cases, $r_d[n]$ is a sequence of independent and identically distributed random variables which follow a uniform distribution.

is zero. Similarly, for any n at which $o_d[n] = 1$, $s_d[n]$ is odd, so the probability that $t_d[n]$ and $t_d[n-1]$ have the same parity is zero. This implies that the successive requantizer is such that $\mathbf{A_e}$ matrices are always even-entries matrices, and $\mathbf{A_o}$ matrices are always odd-entries matrices.

Not only are the $\mathbf{A_e}$ and $\mathbf{A_o}$ matrices for any given successive requantizer even-entries and odd-entries $(2N_t+1)\times(2N_t+1)$ stochastic matrices, respectively, as described above, but the converse is also true: any even-entries and odd-entries $(2N_t+1)\times(2N_t+1)$ stochastic matrices can be used as the $\mathbf{A_e}$ and $\mathbf{A_o}$ matrices, respectively, with which to design a successive requantizer. This is because any such matrices provide a complete description of the pmf of $s_d[n]$ conditioned on $o_d[n]$ and $t_d[n-1]$ at each n, and any chosen pmf can be realized with arbitrarily high accuracy using combinatorial logic elements and a pseudo-random number generator.

It is convenient to define a $(2N_t+1)\times(4N_t+1)$ stochastic matrix that describes the evolution of $s_d[n]$ from times u to $u+m$, with elements given by

$$\Big(\mathbf{S}_{\{o_d[n]\}}[u,m]\Big)(i,j) = \Pr(s_d[u+m] = \mathbf{s}(j) | t_d[u] = \mathbf{t}(i), \; o_d[n]; n = u+1, u+2, \ldots, u+m) \tag{16}$$

where

$$\mathbf{s} = (\, 2N_t \quad 2N_t - 1 \quad \ldots \quad -2N_t \,)^T. \tag{17}$$

As shown in [8], the dependence of the pmf of $s_d[n]$ on $o_d[n]$ implies that, at each time v, $\mathbf{S}_{\{o_d[n]\}}[v-1,1]$ is equal to one of two matrices, denoted as $\mathbf{S_e}$ and $\mathbf{S_o}$. When $o_d[v] = 0$, $\mathbf{S}_{\{o_d[n]\}}[v-1,1] = \mathbf{S_e}$, and when $o_d[v] = 1$, $\mathbf{S}_{\{o_d[n]\}}[v-1,1] = \mathbf{S_o}$. With (8) and (14) this implies that for $m > 1$

$$\mathbf{S}_{\{o_d[n]\}}[u,m] = \mathbf{A}_{\{o_d[n]\}}[u,m-1]\Big(\mathbf{S_e}\,(1-o_d[u+m]) + \mathbf{S_o} o_d[u+m]\Big). \tag{18}$$

Equation (14) implies that each nonzero element in $\mathbf{S_e}$ and $\mathbf{S_o}$ is equal to an element in $\mathbf{A_e}$ and $\mathbf{A_o}$, respectively. Specifically, for i, j, and k such that $\mathbf{s}(k) = \mathbf{t}(j) - \mathbf{t}(i)$, the element in the ith row and kth column of $\mathbf{S_e}$ is equal to that in the ith row and jth column of $\mathbf{A_e}$, and the element in the ith row and kth column of $\mathbf{S_o}$ is equal to that in the ith row and jth column of $\mathbf{A_o}$. Hence, $\mathbf{S_e}$ and $\mathbf{S_o}$ can be deduced from $\mathbf{A_e}$ and $\mathbf{A_o}$ as

$$\mathbf{S_x}(i,j) = \begin{cases} \mathbf{A_x}(i, j+i-2N_t-1), & \text{if } 2N_t + 2 - i \le j \le 4N_t + 2 - i, \\ 0, & \text{if } j \le 2N_t + 1 - i,\ j \ge 4N_t + 3 - i \end{cases} \tag{19}$$

for $\mathbf{x} = \mathbf{e}$ or $\mathbf{o}$.

III. Optimal Quantization in Terms of Immunity to Spurious Tones

A. Theory on Optimal Quantization

The one-step state transition matrices, $\mathbf{A_e}$ and $\mathbf{A_o}$, are said to ensure order-p $t[n]$-convergence if there is a constant b_p such that

$$\lim_{m\to\infty} \mathbf{A}_{\{o_d[n]\}}[u,m]\mathbf{t}^{(p)} = b_p \mathbf{1}_{2N_t+1} \tag{20}$$

for all parity sequences $\{o_d[n],\ d = 0, 1, \ldots, K-1\}$ and any integer u, where

$$\mathbf{t}^{(p)} = \Big({N_t}^p \quad (N_t - 1)^p \quad \ldots \quad (-N_t)^p \Big)^T, \tag{21}$$

$\mathbf{1}_{2N_t+1}$ is a length-$(2N_t+1)$ vector whose elements are all 1, and the convergence of the vector sequence in (18) is exponential.[2] Similarly, they are said to ensure order-q $s[n]$-convergence if there is a constant c_q such that

$$\lim_{m\to\infty} \mathbf{S}_{\{o_d[n]\}}[u,m]\mathbf{s}^{(q)} = c_q \mathbf{1}_{2N_t+1} \tag{22}$$

for all parity sequences $\{o_d[n],\ d = 0, 1, \ldots, K-1\}$ and any integer u, where

$$\mathbf{s}^{(q)} = \Big((2N_t)^q \quad (2N_t - 1)^q \quad \ldots \quad (-2N_t)^q \Big)^T, \tag{23}$$

and the convergence of the vector sequence in (20) is exponential.

Theorems 1 and 2 state sufficient conditions for $t[n]$ and $s[n]$, respectively, to be immune to spurious tones up to any given order.

[2] A length-m vector sequence $\mathbf{b}[0], \mathbf{b}[1], \ldots$ converges exponentially to a vector $\mathbf{b}$ if there exist constants $C \ge 0$ and $0 < \alpha < 1$ such that $|\mathbf{b}[n] - \mathbf{b}| \le C\alpha^n \mathbf{1}_m$ for all integers $n \ge 0$.

Theorem 1: Suppose that $\mathbf{A_e}$ and $\mathbf{A_o}$ ensure order-p $t[n]$-convergence for all positive integers $p \le h_t$, where h_t is a positive integer. Then, $t[n]$ is immune to spurious tones up to order h_t.

Proof: The proof is identical to that of Theorem 1 in [8] except with [8, equation (29)] replaced by

$$\left|E\left\{t^p[n_1]t^p[n_2] - C_{t^p}\right\}\right| \le D_1\alpha^{|n_2-n_1|} + D_2\alpha^{\min\{n_1,n_2\}}, \tag{24}$$

for some positive constants D_1 and D_2 and a constant $0 < \alpha < 1$, [8, equation (31)] replaced by

$$\begin{aligned} |J_{2,1}| &\le \frac{1}{L}\sum_{\substack{n_1=0\\ n_1\neq n_2}}^{L-1}\sum_{n_2=0}^{L-1}\left(D_1\alpha^{|n_1-n_2|} + D_2\alpha^{\min\{n_1,n_2\}}\right)\\ &\le \frac{D_1}{L}\sum_{n_1=0}^{L-1}\sum_{n_2=0}^{L-1}\alpha^{|n_1-n_2|}\\ &+\frac{D_2}{L}\left(\sum_{0\le n_1<n_2\le L-1}\alpha^{n_1} + \sum_{0\le n_2<n_1\le L-1}\alpha^{n_2}\right)\\ &\le \frac{D_1}{L}\sum_{n_1=0}^{L-1}\left(2\sum_{n=0}^{L-1}\alpha^n\right) + D_2\left(2\sum_{n=0}^{L-1}\alpha^n\right)\\ &\le 2\left(D_1+D_2\right)\frac{1-\alpha^L}{1-\alpha} \le 2\left(D_1+D_2\right)\frac{1}{1-\alpha}, \end{aligned} \tag{25}$$

and Lemma 1 in [8] replaced by Lemma 1 in the Appendix of this paper. ■

Theorem 2: Suppose that $\mathbf{A_e}$ and $\mathbf{A_o}$ ensure order-q $s[n]$-convergence for all positive integers $q \le h_s$, where h_s is a positive integer. Then, $s[n]$ is immune to spurious tones up to order h_s.

Proof: The proof is identical to that of Theorem 2 in [8] except with p replaced by q, [8, equation (36)] replaced by

$$\left|E\left\{s^q[n_1]s^q[n_2] - C_{s^q}\right\}\right| \le E_1\beta^{|n_2-n_1|} + E_2\beta^{\min\{n_1,n_2\}}, \tag{26}$$

for some positive constants E_1 and E_2 and a constant $0 < \beta < 1$, and Lemma 2 in [8] replaced by Lemma 2 in the Appendix of this paper. ■

Theorem 3 provides sufficient conditions on $\mathbf{A_e}$ and $\mathbf{A_o}$ for $t[n]$ to be immune to spurious tones up to order $2N_t - 1$.

Theorem 3: Let $\mathbf{A_e}$ and $\mathbf{A_o}$ be $(2N_t+1)\times(2N_t+1)$ matrices with elements that satisfy

$$\mathbf{A_e}(i,j) = \begin{cases} \frac{1}{2^{2N_t-1}}\binom{2N_t}{j-1} + \left(\mathbf{Q}^T\mathbf{P}\mathbf{Q}\right)(i,j), & \text{if } i+j \text{ is even},\\ 0, & \text{if } i+j \text{ is odd}, \end{cases} \tag{27}$$

$$\mathbf{A_o}(i,j) = \begin{cases} 0, & \text{if } i+j \text{ is even},\\ \frac{1}{2^{2N_t-1}}\binom{2N_t}{j-1} + \left(\mathbf{Q}^T\mathbf{P}\mathbf{Q}\right)(i,j), & \text{if } i+j \text{ is odd}, \end{cases} \tag{28}$$

where N_t is any integer greater than 1, $\mathbf{Q}$ is the $N_t \times (2N_t+1)$ matrix

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -1\\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & -1 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & 1 & 0 & -1 & \dots & 0 & 0 \end{pmatrix} \tag{29}$$

and $\mathbf{P}$ is any $N_t \times N_t$ matrix whose elements satisfy

$$|\mathbf{P}(i,j)| \le \frac{1}{2^{2N_t-1}}\binom{2N_t}{j-1} \tag{30}$$

and for each row i

$$|\mathbf{P}(i,j)| \neq \frac{1}{2^{2N_t-1}}\binom{2N_t}{j-1} \tag{31}$$

for at least one $j \in \{2,4,\dots,N_t\}$ if N_t is even and one $j \in \{1,3,\dots,N_t\}$ if N_t is odd. Then, $t[n]$ is immune to spurious tones up to order $2N_t - 1$.

Note that the $\mathbf{A_e}$ and $\mathbf{A_o}$ matrices given by (13) satisfy the conditions of Theorem 3. Specifically, (25) and (26) with

$$\mathbf{P} = \begin{pmatrix} 1/8 & 1/4\\ 1/16 & 1/8 \end{pmatrix} \tag{32}$$

yield the $\mathbf{A_e}$ and $\mathbf{A_o}$ matrices given by (13).

Proof of Theorem 3: It is first shown that $\mathbf{A_e}$ and $\mathbf{A_o}$ are valid one-step state transition matrices for the successive requantizer, i.e., that they are stochastic even-entries and odd-entries matrices, respectively. These results are then used to show that Theorem 1 holds for $h_t = 2N_t - 1$, which completes the proof.

By definition, $\mathbf{A_e}$ and $\mathbf{A_o}$ are even-entries and odd-entries matrices, respectively. To show that they are stochastic matrices, it is sufficient to show that all their elements are nonnegative and that the sum of the elements on each row of each matrix is 1.

It follows from (25) and (26) that a sufficient condition for the elements of $\mathbf{A_e}$ and $\mathbf{A_o}$ to be nonnegative is

$$\left|\left(\mathbf{Q}^T\mathbf{P}\mathbf{Q}\right)(i,j)\right| \le \frac{1}{2^{2N_t-1}}\binom{2N_t}{j-1} \tag{33}$$

for all $i, j \in \{1,2,\dots,2N_t+1\}$. The matrix $\mathbf{Q}$ can be written as

$$\mathbf{Q} = \left(\mathbf{I}_{N_t}, \mathbf{0}_{N_t}, -\mathbf{J}_{N_t}\right), \tag{34}$$

where $\mathbf{I}_{N_t}$ is the $N_t \times N_t$ identity matrix, $\mathbf{0}_{N_t}$ is a length-N_t vector whose elements are all 0, and $\mathbf{J}_{N_t}$ is the $N_t \times N_t$ *exchange matrix*, i.e., the $N_t \times N_t$ matrix for which all the elements in the anti-diagonal are 1 and all other elements are 0. Thus,

$$\mathbf{Q}^T\mathbf{P}\mathbf{Q} = \begin{pmatrix} \mathbf{P} & \mathbf{0}_{N_t} & -\mathbf{P}\mathbf{J}_{N_t}\\ \mathbf{0}_{N_t}{}^T & 0 & \mathbf{0}_{N_t}{}^T\\ -\mathbf{J}_{N_t}\mathbf{P} & \mathbf{0}_{N_t} & \mathbf{J}_{N_t}\mathbf{P}\mathbf{J}_{N_t} \end{pmatrix}. \tag{35}$$

The definition of exchange matrices implies that

$$\left(\mathbf{P}\mathbf{J}_{N_t}\right)(i,j) = \mathbf{P}(i, N_t+1-j), \tag{36}$$

$$(\mathbf{J}_{N_t}\mathbf{P})(i,j) = \mathbf{P}(N_t+1-i,j), \text{ and} \tag{37}$$
$$(\mathbf{J}_{N_t}\mathbf{P}\mathbf{J}_{N_t})(i,j) = \mathbf{P}(N_t+1-i,N_t+1-j) \tag{38}$$

for all $i, j \in \{1,2,\ldots,N_t\}$. Combining (35) and (36) yields

$$(\mathbf{J}_{N_t}\mathbf{P}\mathbf{J}_{N_t})(i,j) = (\mathbf{J}_{N_t}\mathbf{P})(i,N_t+1-j), \tag{39}$$

which with (33) and (34) implies

$$\left(\mathbf{Q}^T\mathbf{P}\mathbf{Q}\right)(i,j) = -\left(\mathbf{Q}^T\mathbf{P}\mathbf{Q}\right)(i,2N_t+2-j) \tag{40}$$

for all $i, j \in \{1,2,\ldots,2N_t+1\}$. It follows from (28), (33), and (35) that (31) is satisfied for all $j \in \{1,2,\ldots,N_t\}$ and $i \in \{1,2,\ldots,2N_t+1\}$. This, with (38), implies that

$$\left|\left(\mathbf{Q}^T\mathbf{P}\mathbf{Q}\right)(i,j)\right| \le \begin{cases} \dfrac{1}{2^{2N_t-1}}\dbinom{2N_t}{j-1}, & \text{if } 1 \le j \le N_t+1, \\ \dfrac{1}{2^{2N_t-1}}\dbinom{2N_t}{2N_t+1-j}, & \text{if } N_t+2 \le j \le 2N_t+1 \end{cases} \tag{41}$$

for all $i \in \{1,2,\ldots,2N_t+1\}$. Binomial coefficients have the property that

$$\binom{2N_t}{j-1} = \binom{2N_t}{2N_t+1-j} \tag{42}$$

for all $j \in \{1,2,\ldots,2N_t+1\}$, so (39) is equivalent to (31) for all $i, j \in \{1,2,\ldots,2N_t+1\}$. This completes the proof that the elements of $\mathbf{A_e}$ and $\mathbf{A_o}$ are nonnegative.

To show that $\mathbf{A_e}$ and $\mathbf{A_o}$ are stochastic matrices, it remains to show that the sum of the elements in each of their rows is 1. It follows from (25) and (26) that the sum of the elements on the ith row of $\mathbf{A_e}$ or $\mathbf{A_o}$ can either be written as

$$\frac{1}{2^{2N_t-1}}\sum_{j=1,j \text{ odd}}^{2N_t+1}\binom{2N_t}{j-1} + \sum_{j=1,j \text{ odd}}^{2N_t+1}\left(\mathbf{Q}^T\mathbf{P}\mathbf{Q}\right)(i,j) \text{ or} \tag{43}$$

$$\frac{1}{2^{2N_t-1}}\sum_{j=2,j \text{ even}}^{2N_t}\binom{2N_t}{j-1} + \sum_{j=2,j \text{ even}}^{2N_t}\left(\mathbf{Q}^T\mathbf{P}\mathbf{Q}\right)(i,j). \tag{44}$$

It follows from (38) that the second sum in each of (41) and (42) is 0. The first sums in (41) and (42) can be rewritten as

$$\frac{1}{2^{2N_t}}\left[\sum_{j=1}^{2N_t+1}\binom{2N_t}{j-1} - \sum_{j=1}^{2N_t+1}\binom{2N_t}{j-1}(-1)^j\right] \text{ and} \tag{45}$$

$$\frac{1}{2^{2N_t}}\left[\sum_{j=1}^{2N_t+1}\binom{2N_t}{j-1} + \sum_{j=1}^{2N_t+1}\binom{2N_t}{j-1}(-1)^j\right], \tag{46}$$

respectively. The Binomial Theorem implies that the first and second sums in each of (43) and (44) equal $(1+1)^{2N_t}$ and $(1-1)^{2N_t}$, respectively. Thus, (43) and (44) each evaluate to 1, so the sum of the elements on each row of $\mathbf{A_e}$ and $\mathbf{A_o}$ is 1.

To complete the proof of the theorem it is sufficient to prove that $\mathbf{A_e}$ and $\mathbf{A_o}$ ensure order-p $t[n]$-convergence for all positive integers $p \le 2N_t - 1$ so that Theorem 1 can be applied. This is done in two parts. First, it is shown that $\mathbf{A_e}$ and $\mathbf{A_o}$ are centrosymmetric [3], that all their even-entries row vectors have at least $1 + \lfloor N_t/2 \rfloor$ nonzero entries, and that all their odd-entries row vectors have at least $1 + \lfloor (N_t+1)/2 \rfloor$ nonzero entries.[4] With Lemma 3 in the Appendix, this shows that $\mathbf{A_e}$ and $\mathbf{A_o}$ ensure order-p $t[n]$-convergence for all odd positive integers $p \le 2N_t - 1$. Second, it is shown that for each even positive integer $p \le 2N_t - 1$

$$\mathbf{A_e}\mathbf{t}^{(p)} = \mathbf{A_o}\mathbf{t}^{(p)} = b_p\mathbf{1}_{2N_t+1} \tag{47}$$

for some constant b_p. With Lemma 4 in the Appendix, this shows that $\mathbf{A_e}$ and $\mathbf{A_o}$ ensure order-p $t[n]$-convergence for all even positive integers $p \le 2N_t - 1$.

Combining (34) and (35) yields

$$(\mathbf{P}\mathbf{J}_{N_t})(i,j) = (\mathbf{J}_{N_t}\mathbf{P})(N_t+1-i,N_t+1-j) \tag{48}$$

for all $i, j \in \{1,2,\ldots,N_t\}$. This, with (33) and (36), implies that

$$\left(\mathbf{Q}^T\mathbf{P}\mathbf{Q}\right)(i,j) = \left(\mathbf{Q}^T\mathbf{P}\mathbf{Q}\right)(2N_t+2-i,2N_t+2-j) \tag{49}$$

for all $i, j \in \{1,2,\ldots,2N_t+1\}$. It follows from (40) and (47) that

$$\frac{1}{2^{2N_t-1}}\binom{2N_t}{j-1} + \left(\mathbf{Q}^T\mathbf{P}\mathbf{Q}\right)(i,j) = \frac{1}{2^{2N_t-1}}\binom{2N_t}{(2N_t+2-j)-1} + \left(\mathbf{Q}^T\mathbf{P}\mathbf{Q}\right)(2N_t+2-i,2N_t+2-j) \tag{50}$$

for all $i, j \in \{1,2,\ldots,2N_t+1\}$. This, with (25) and (26), implies that $\mathbf{A_e}$ and $\mathbf{A_o}$ are centrosymmetric.

Let $\mathbf{v_o}$ be any odd-entries row vector of either $\mathbf{A_e}$ or $\mathbf{A_o}$. It follows from (25) and (26) that there is a value of $i \in \{1,2,\ldots,2N_t+1\}$ such that the elements of $\mathbf{v_o}$ can be written as

$$\mathbf{v_o}(j) = \begin{cases} \dfrac{1}{2^{2N_t-1}}\dbinom{2N_t}{j-1} + \left(\mathbf{Q}^T\mathbf{P}\mathbf{Q}\right)(i,j), & \text{if } j = 1,\ 3,\ \ldots, \text{or } 2N_t+1, \\ 0, & \text{if } j = 2,\ 4,\ \ldots, \text{or } 2N_t. \end{cases} \tag{51}$$

It follows from (40) that

$$\frac{1}{2^{2N_t-1}}\binom{2N_t}{N_t+1-k} = \frac{1}{2^{2N_t-1}}\binom{2N_t}{N_t+1+k} \tag{52}$$

and from (38) that

$$\left(\mathbf{Q}^T\mathbf{P}\mathbf{Q}\right)(i,N_t+1-k) = -\left(\mathbf{Q}^T\mathbf{P}\mathbf{Q}\right)(i,N_t+1+k) \tag{53}$$

for each $k \in \{1,2,\ldots,N_t\}$. Therefore, it is not possible for (49) to be zero for both $j = N_t+1-k$ and $j = N_t+1+k$ for any $k \in \{1,3,\ldots,N_t\}$ if N_t is odd or any $k \in \{2,4,\ldots,N_t\}$ if N_t is even. As indicated by (33), $(\mathbf{Q}^T\mathbf{P}\mathbf{Q})(i,N_t+1) = 0$, so if

[3]An $N \times M$ matrix $\mathbf{A}$ is said to be centrosymmetric if $\mathbf{A}(i,j) = \mathbf{A}(N+1-i,M+1-j)$ for all $i \in \{1,2,\ldots,N\}$ and $j \in \{1,2,\ldots,M\}$.

[4]For any number x, $\lfloor x \rfloor$ denotes the largest integer not greater than x.

N_t is even then $\mathbf{v_o}(N_t+1)$ is nonzero. Equations (33), (35), and (51) with the theorem's stated conditions under which (29) holds imply that if N_t is odd there is a value of $k \in \{1, 3, \ldots, N_t\}$ for which (49) is nonzero for both $j = N_t + 1 - k$ and $j = N_t + 1 + k$. These results imply that $\mathbf{v_o}$ has at least $1 + N_t/2$ nonzero elements if N_t is even and at least $1 + (N_t + 1)/2$ nonzero elements if N_t is odd, or, equivalently, that $\mathbf{v_o}$ has at least $1 + \lfloor (N_t+1)/2 \rfloor$ nonzero elements regardless of whether N_t is even or odd. Almost identical reasoning leads to the conclusion that each even-entries row vector of either $\mathbf{A_e}$ or $\mathbf{A_o}$ has at least $1 + \lfloor N_t/2 \rfloor$ nonzero elements.

Suppose p is even. It follows from (9), (25), and (26) that the ith element of $\mathbf{A_e t}^{(p)}$ or $\mathbf{A_o t}^{(p)}$ can be written as

$$\sum_{j=2, j \text{ even}}^{2N_t} (N_t+1-j)^p \cdot \left(\frac{1}{2^{2N_t-1}} \binom{2N_t}{j-1} + \left(\mathbf{Q}^T\mathbf{PQ}\right)(i,j) \right) \tag{54}$$

or

$$\sum_{j=1, j \text{ odd}}^{2N_t+1} (N_t+1-j)^p \cdot \left(\frac{1}{2^{2N_t-1}} \binom{2N_t}{j-1} + \left(\mathbf{Q}^T\mathbf{PQ}\right)(i,j) \right). \tag{55}$$

Given that

$$(N_t + 1 - j)^p = (N_t + 1 - (2N_t + 2 - j))^p \tag{56}$$

for all $j \in \{1, 2, \ldots, 2N_t + 1\}$, (52) and (53) can be rewritten as

$$\begin{aligned} \frac{1}{2^{2N_t-1}} &\sum_{j=2, j \text{ even}}^{2N_t} (N_t + 1 - j)^p \binom{2N_t}{j-1} \\ &+ \sum_{j=2, j \text{ even}}^{N_t} (N_t + 1 - j)^p \Big[\left(\mathbf{Q}^T\mathbf{PQ}\right)(i,j) \\ &\qquad + \left(\mathbf{Q}^T\mathbf{PQ}\right)(i, 2N_t + 2 - j) \Big] \end{aligned} \tag{57}$$

and

$$\begin{aligned} \frac{1}{2^{2N_t-1}} &\sum_{j=1, j \text{ odd}}^{2N_t+1} (N_t + 1 - j)^p \binom{2N_t}{j-1} \\ &+ \sum_{j=1, j \text{ odd}}^{N_t} (N_t + 1 - j)^p \Big[\left(\mathbf{Q}^T\mathbf{PQ}\right)(i,j) \\ &\qquad + \left(\mathbf{Q}^T\mathbf{PQ}\right)(i, 2N_t + 2 - j) \Big], \end{aligned} \tag{58}$$

respectively. It follows from (38) that the second sums in (55) and (56) equal 0. Therefore, subtracting (55) from (56) yields

$$\frac{1}{2^{2N_t-1}} \sum_{j=1}^{2N_t+1} (N_t + 1 - j)^p (-1)^j \binom{2N_t}{j-1}. \tag{59}$$

The expression in (57) is 0 for each $p \in \{2, 4, \ldots, 2N_t - 2\}$ [16]. Thus, for each such p, (52) and (53) are equal, so there exists a value b_p such that (45) holds. ■

Theorem 4 proves that Theorem 2 cannot hold for $h_s = 4N_t - 2$, although as shown by example in the next section it can hold for $h_s = 4N_t - 3$.

Theorem 4: There do not exist $\mathbf{A_e}$ and $\mathbf{A_o}$ matrices such that Theorem 2 holds for $h_s = 4N_t - 2$.

Proof: The proof is by contradiction. Suppose Theorem 2 holds for $h_s = 4N_t - 2$. Let u be any integer and $o_d[n]$ be a parity sequence that satisfies

$$o_d[u + m] = \begin{cases} 0, & \text{if } m \text{ is even,} \\ 1, & \text{if } m \text{ is odd} \end{cases} \tag{60}$$

for all positive integers m. By Lemma 5, $\mathbf{A}_{\{o_d[n]\}}[u, m]$ is either an even-entries or an odd-entries matrix for each positive integer m, so its row vectors alternate between even-entries and odd-entries vectors. For each m, let $\mathbf{v_e}[m]$ be an even-entries row vector in $\mathbf{A}_{\{o_d[n]\}}[u, m]$. It is first shown that

$$\lim_{m\to\infty} \sum_{j=2}^{4N_t} \left(\mathbf{v_e}[2m]\mathbf{S_o} - \mathbf{v_e}[2m-1]\mathbf{S_e}\right)(j) \cdot (2N_t+1-j)^q = 0 \tag{61}$$

for all $q \in \{0, 1, \ldots, 4N_t - 2\}$. This is then used to prove that

$$\lim_{m\to\infty} \left(\mathbf{v_e}[2m]\mathbf{S_o} - \mathbf{v_e}[2m-1]\mathbf{S_e}\right) = \mathbf{0}_{4N_t+1}{}^T. \tag{62}$$

Finally, it is shown that

$$\lim_{m\to\infty} \left(\mathbf{v_e}[2m]\mathbf{S_o} - \mathbf{v_e}[2m-1]\mathbf{S_e}\right) \cdot \begin{pmatrix} 0 & 1 & 0 & 1 & \ldots & 0 \end{pmatrix}^T = 1. \tag{63}$$

However, (60) implies that

$$\lim_{m\to\infty} \left(\mathbf{v_e}[2m]\mathbf{S_o} - \mathbf{v_e}[2m-1]\mathbf{S_e}\right) \cdot \begin{pmatrix} 0 & 1 & 0 & 1 & \ldots & 0 \end{pmatrix}^T = 0, \tag{64}$$

which contradicts (61), so Theorem 2 must not hold for $h_s = 4N_t - 2$.

By assumption, $\mathbf{A_e}$ and $\mathbf{A_o}$ ensure order-q $s[n]$-convergence for each positive integer $q \leq 4N_t - 2$. With (16), this implies that, for each such q,

$$\lim_{m\to\infty} \mathbf{A}_{\{o_d[n]\}}[u, m-1] \left(\mathbf{S_e}\left(1 - o_d[u+m]\right) + \mathbf{S_o} o_d[u+m]\right) \mathbf{s}^{(q)} = c_q \mathbf{1}_{2N_t+1} \tag{65}$$

for some constant c_q. Given (58), (63) implies that

$$\lim_{m\to\infty} \mathbf{A}_{\{o_d[n]\}}[u, 2m-1]\mathbf{S_e}\mathbf{s}^{(q)} = c_q \mathbf{1}_{2N_t+1} \text{ and} \tag{66}$$

$$\lim_{m\to\infty} \mathbf{A}_{\{o_d[n]\}}[u, 2m]\mathbf{S_o}\mathbf{s}^{(q)} = c_q \mathbf{1}_{2N_t+1}, \tag{67}$$

so

$$\lim_{m\to\infty} \mathbf{v_e}[2m-1]\mathbf{S_e}\mathbf{s}^{(q)} = c_q \text{ and} \tag{68}$$

$$\lim_{m\to\infty} \mathbf{v_e}[2m]\mathbf{S_o}\mathbf{s}^{(q)} = c_q. \tag{69}$$

Therefore, for $q \in \{1, 2, \ldots, 4N_t - 2\}$,

$$\lim_{m\to\infty} \left(\mathbf{v_e}[2m]\mathbf{S_o} - \mathbf{v_e}[2m-1]\mathbf{S_e}\right)\mathbf{s}^{(q)} = 0, \tag{70}$$

or, equivalently,

$$\lim_{m\to\infty} \sum_{j=1}^{4N_t+1} \left(\mathbf{v_e}[2m]\mathbf{S_o} - \mathbf{v_e}[2m-1]\mathbf{S_e}\right)(j) \cdot \mathbf{s}^{(q)}(j) = 0. \tag{71}$$

Since $\mathbf{S_e}$ and $\mathbf{S_o}$ are stochastic matrices,

$$\mathbf{S_e}\mathbf{1}_{4N_t+1} = \mathbf{S_o}\mathbf{1}_{4N_t+1} = \mathbf{1}_{2N_t+1}. \tag{72}$$

Given that $\mathbf{A}_{\{o_d[n]\}}[u,m]$ is a stochastic matrix for each m, this implies that

$$\begin{aligned}\lim_{m\to\infty} &\mathbf{A}_{\{o_d[n]\}}[u,m]\mathbf{S_e}\mathbf{1}_{4N_t+1}\\ &= \lim_{m\to\infty} \mathbf{A}_{\{o_d[n]\}}[u,m]\mathbf{S_o}\mathbf{1}_{4N_t+1}\\ &= \lim_{m\to\infty} \mathbf{A}_{\{o_d[n]\}}[u,m]\mathbf{1}_{2N_t+1} = \mathbf{1}_{2N_t+1},\end{aligned} \tag{73}$$

so (64) and (65) and, consequently, (69) also hold for $q = 0$.

Note that if $j = 1$, (17) implies that $\mathbf{S_e}(i,j) = \mathbf{S_o}(i,j) = 0$ for all i except $i = 2N_t + 1$, and if $j = 4N_t + 1$, (17) implies that $\mathbf{S_e}(i,j) = \mathbf{S_o}(i,j) = 0$ for all i except $i = 1$. Additionally, since, for each m, $\mathbf{v_e}[m]$ is an even-entries vector, $(\mathbf{v_e}[m])(1) = (\mathbf{v_e}[m])(2N_t+1) = 0$. Thus, for $j \in \{1, 4N_t + 1\}$ and each m,

$$\left(\mathbf{v_e}[2m]\mathbf{S_o} - \mathbf{v_e}[2m-1]\mathbf{S_e}\right)(j) = 0. \tag{74}$$

This implies that (69) can be rewritten as

$$\lim_{m\to\infty} \sum_{j=2}^{4N_t} \left(\mathbf{v_e}[2m]\mathbf{S_o} - \mathbf{v_e}[2m-1]\mathbf{S_e}\right)(j) \cdot \mathbf{s}^{(q)}(j) = 0. \tag{75}$$

This, with (15), implies that (59) holds for $q \in \{0, 1, \ldots, 4N_t - 2\}$.

Equation (59) for all $q \in \{0, 1, \ldots, 4N_t - 2\}$ can be written in matrix form as

$$\lim_{m\to\infty} \mathbf{x}[m] \cdot \mathbf{M} = \mathbf{0}_{4N_t-1}{}^T, \tag{76}$$

where $\mathbf{x}[m]$ is the length-$(4N_t - 1)$ subvector of $(\mathbf{v_e}[2m]\mathbf{S_o} - \mathbf{v_e}[2m-1]\mathbf{S_e})$ formed by rows 2 through $4N_t$, i.e.,

$$(\mathbf{x}[m])(j) = \left(\mathbf{v_e}[2m]\mathbf{S_o} - \mathbf{v_e}[2m-1]\mathbf{S_e}\right)(j+1) \tag{77}$$

for $1 \le j \le 4N_t - 1$, and $\mathbf{M}$ is the $(4N_t - 1) \times (4N_t - 1)$ matrix

$$\begin{pmatrix} 1 & (2N_t-1)^1 & (2N_t-1)^2 & \ldots & (2N_t-1)^{4N_t-2}\\ 1 & (2N_t-2)^1 & (2N_t-2)^2 & \ldots & (2N_t-2)^{4N_t-2}\\ 1 & (2N_t-3)^1 & (2N_t-3)^2 & \ldots & (2N_t-3)^{4N_t-2}\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ 1 & (-2N_t+1)^1 & (-2N_t+1)^2 & \ldots & (-2N_t+1)^{4N_t-2} \end{pmatrix}. \tag{78}$$

Matrix $\mathbf{M}$ is a square Vandermonde matrix [17]. No two elements in the second column of $\mathbf{M}$ are equal to each other, so it follows from the properties of Vandermonde matrices that the determinant of $\mathbf{M}$ is nonzero, which implies that $\mathbf{M}$ is invertible. Right multiplying both sides of (74) by the inverse of $\mathbf{M}$ yields

$$\lim_{m\to\infty} \mathbf{x}[m] = \mathbf{0}_{4N_t-1}{}^T. \tag{79}$$

Since (72) holds for $j \in \{1, 4N_t + 1\}$, (77) implies that (60) holds.

Given that $\mathbf{A_e}$ is an even-entries matrix, $\mathbf{A_e}(i, j + i - 2N_t - 1) = 0$ whenever j is even, because $i + (j + i - 2N_t - 1)$ is odd. Therefore, (17) implies that $\mathbf{S_e}(i,j) = 0$ whenever j is even. In particular, it follows that the sum of the even-indexed entries in each row of $\mathbf{S_e}$ is 0. Given that $\mathbf{A_o}$ is an odd-entries matrix, $\mathbf{A_o}(i, j + i - 2N_t - 1) = 0$ whenever j is odd, because $i + (j + i - 2N_t - 1)$ is even. Therefore, (17) implies that $\mathbf{S_o}(i,j) = 0$ whenever j is odd, so the even-indexed entries in each row of $\mathbf{S_o}$ include all of the row's non-zero entries. It follows that the sum of these entries must be unity because $\mathbf{S_o}$ is a stochastic matrix. These results imply that

$$\begin{aligned}\lim_{m\to\infty} &\left(\mathbf{v_e}[2m]\mathbf{S_o} - \mathbf{v_e}[2m-1]\mathbf{S_e}\right) \cdot \begin{pmatrix}0 & 1 & 0 & 1 & \ldots & 0\end{pmatrix}^T\\ &= \lim_{m\to\infty} \mathbf{v_e}[2m]\mathbf{1}_{2N_t+1}.\end{aligned} \tag{80}$$

This is equivalent to (61), because $\mathbf{v_e}[m]$ is a row vector of a stochastic matrix. ■

B. Optimal Successive Requantizers

Given any quantizer, let o_t and o_s denote the orders up to which $t[n]$ and $s[n]$, respectively, are immune to spurious tones, and let N_t denote the smallest integer for which $|t[n]| \le N_t$ over all n. The design strategy for the successive requantizer presented in this paper is to find $\mathbf{A_e}$ and $\mathbf{A_o}$ matrices that maximize the values of o_t and o_s.

The results of [14] prove that $o_t \le 2N_t - 1$ regardless of the quantizer used. Theorem 3 shows that for a successive requantizer, there exist $\mathbf{A_e}$ and $\mathbf{A_o}$ matrices that ensure $o_t = 2N_t - 1$. Thus, successive requantizers are optimal quantizers in terms of the order up to which $t[n]$ can be immune to spurious tones.

In this paper, $\mathbf{A_e}$ and $\mathbf{A_o}$ matrices for $N_t = 2, 3, 4$, and 5 are presented for which $o_t = 2N_t - 1$ and which satisfy Theorem 2 for $h_s = 4N_t - 3$, which implies that $o_s = 4N_t - 3$. As proven by Theorem 4, $4N_t - 3$ is the maximum value of h_s for which Theorem 2 can hold. Thus, the matrices presented are optimal in the sense that o_t and h_s are as large as possible for the corresponding values of N_t.

The procedure to find the $\mathbf{A_e}$ and $\mathbf{A_o}$ matrices is to use (25)–(27) and find the elements of matrix $\mathbf{P}$ by solving the system of equations

$$\begin{aligned}\mathbf{A_e}\mathbf{S_e}\mathbf{s}^{(q)} = \mathbf{A_e}\mathbf{S_o}\mathbf{s}^{(q)} &= \mathbf{A_o}\mathbf{S_e}\mathbf{s}^{(q)}\\ = \mathbf{A_o}\mathbf{S_o}\mathbf{s}^{(q)} &= c_q\mathbf{1}_{2N_t+1},\end{aligned} \tag{81}$$

with $\mathbf{S_e}$ and $\mathbf{S_o}$ given by (17), for all even positive integers $q \le 4N_t - 4$ and any constants $c_2, c_4, \ldots, c_{4N_t-4}$, following

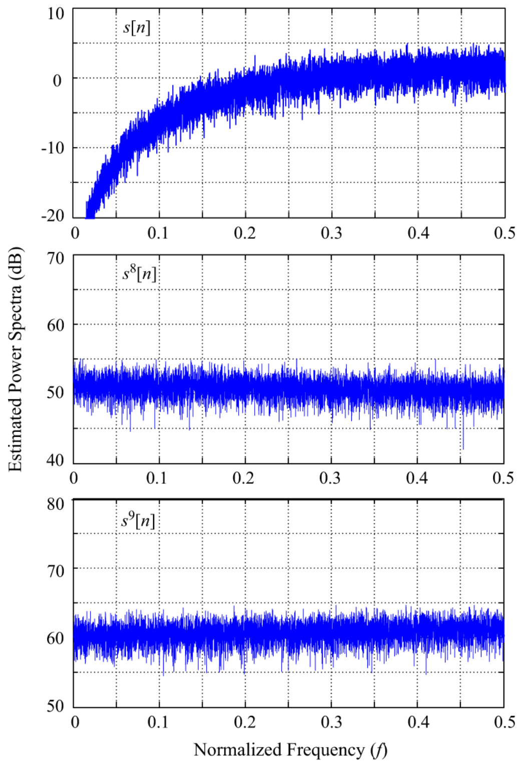

Fig. 5. Estimated power spectra of the quantization noise of a simulated successive requantizer for which $N_t = 3$ before and after the application of 8^{th} and 9^{th} order nonlinear distortion.

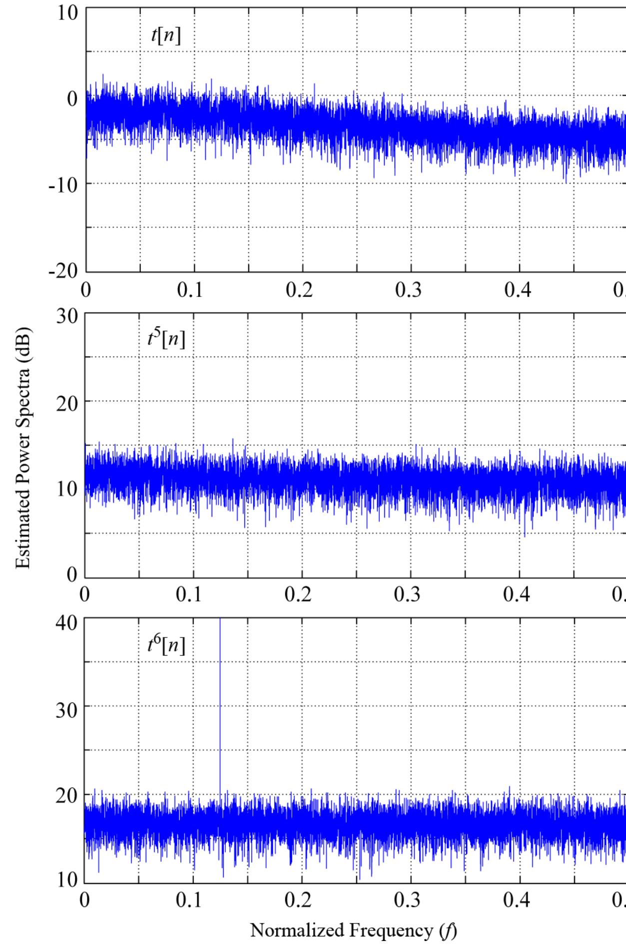

Fig. 6. Estimated power spectra of the running sum of the quantization noise of a simulated successive requantizer for which $N_t = 3$ before and after the application of 5^{th} and 6^{th} order nonlinear distortion.

the constraints specified in Theorem 3. By Theorem 3, these $\mathbf{A_e}$ and $\mathbf{A_o}$ matrices guarantee that $o_t = 2N_t - 1$. As shown in the proof of Theorem 3, $\mathbf{A_e}$ and $\mathbf{A_o}$ satisfy the conditions of Lemma 3 in the Appendix, so $\mathbf{A_e}$ and $\mathbf{A_o}$ ensure order-q $s[n]$-convergence for all odd q. Additionally, since the system of equations specified by (79) holds, it follows from Lemma 6 in the Appendix that $\mathbf{A_e}$ and $\mathbf{A_o}$ ensure order-q $s[n]$-convergence for all even positive integers $q \le 4N_t - 4$. Thus, $h_s = 4N_t - 3$.

For each N_t, the system of equations specified by (79) was solved using Matlab's solve() function [18]. For $N_t = 2$, the $\mathbf{A_e}$ and $\mathbf{A_o}$ matrices found are those presented in [8] and given by (13). For each $N_t = 3$, 4, and 5, $\mathbf{P}_{N_t}$, i.e., the $\mathbf{P}$ matrix from which $\mathbf{A_e}$ and $\mathbf{A_o}$ can be constructed using (25)–(27), is

$$\mathbf{P}_3 = \begin{pmatrix} \frac{1}{32} & \frac{1}{8} & \frac{1}{32} \\ \frac{1}{48} & \frac{1}{12} & \frac{5}{48} \\ \frac{3}{160} & \frac{1}{24} & \frac{5}{96} \end{pmatrix}, \tag{82}$$

$$\mathbf{P}_4 = \begin{pmatrix} \frac{1}{128} & \frac{3}{64} & \frac{63}{400} & \frac{21}{64} \\ \frac{3}{512} & \frac{9}{256} & \frac{7}{128} & -\frac{63}{640} \\ \frac{7}{3200} & \frac{1}{64} & \frac{7}{128} & \frac{1}{16} \\ \frac{3}{512} & \frac{3}{80} & \frac{1}{32} & \frac{7}{256} \end{pmatrix}, \text{ and} \tag{83}$$

$$\mathbf{P}_5 = \begin{pmatrix} \frac{1}{512} & \frac{1}{64} & \frac{355}{6083} & -\frac{3}{640} & -\frac{458}{1193} \\ \frac{1}{640} & \frac{1}{80} & \frac{51}{1000} & \frac{39}{1600} & \frac{337}{2122} \\ \frac{67}{64000} & \frac{37}{3200} & \frac{380}{7179} & \frac{177}{8053} & \frac{139}{6736} \\ \frac{3}{64000} & \frac{67}{6400} & \frac{101}{7578} & \frac{3}{80} & \frac{51}{1280} \\ -\frac{1}{512} & \frac{67}{6400} & \frac{31}{2560} & \frac{41}{1600} & \frac{104}{8781} \end{pmatrix}, \tag{84}$$

respectively.

The immunity to spurious tones achieved for each N_t suggests that o_s can be increased by increasing N_t, but this result has yet to be proven theoretically for arbitrary values of N_t.

A quantization noise sequence, $s[n]$, was generated by simulating a successive requantizer with $N_t = 3$, wherein $\mathbf{A_e}$ and $\mathbf{A_o}$ are constructed from the $\mathbf{P}_3$ matrix in (82). Fig. 5 shows estimated power spectra of $s[n]$ before and after the application of 8^{th} and 9^{th} order distortion, and Fig. 6 shows estimated power spectra of the running sum of the quantization noise $t[n]$ before and after the application of 5^{th} and 6^{th} order distortion. As expected, the power spectra of $s^p[n]$ for $p = 1$, 8, and 9 and $t^p[n]$ for $p = 1$ and 5 show no visible spurious tones, as

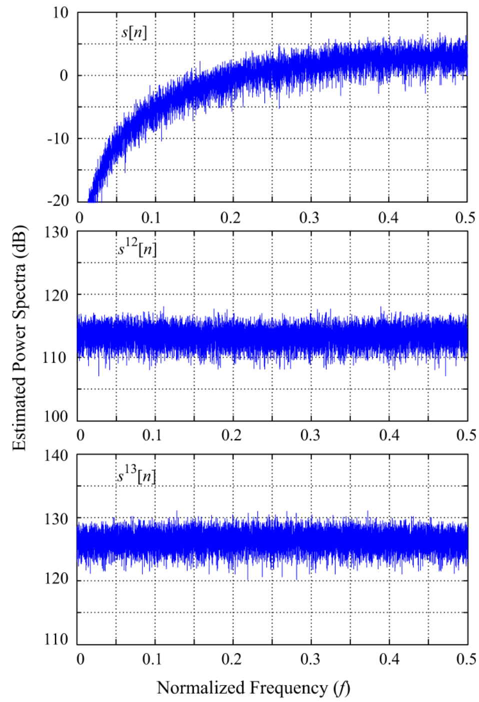

Fig. 7. Estimated power spectra of the quantization noise of a simulated successive requantizer for which $N_t = 4$ before and after the application of 12^{th} and 13^{th} order nonlinear distortion.

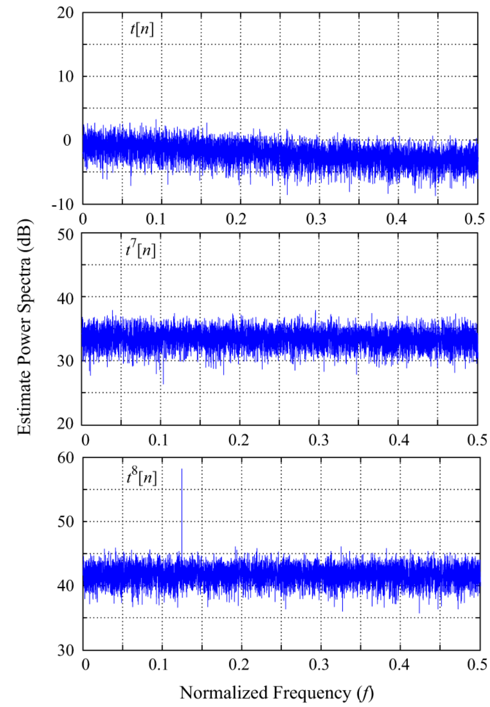

Fig. 8. Estimated power spectra of the running sum of the quantization noise of a simulated successive requantizer for which $N_t = 4$ before and after the application of 7^{th} and 8^{th} order nonlinear distortion.

$o_s = 4N_t - 3 = 9$ and $o_t = 2N_t - 1 = 5$, while that of $t^6[n]$ shows spurious tones. Similarly, a quantization noise sequence, $s[n]$, was generated by simulating a successive requantizer with $N_t = 4$, wherein $\mathbf{A_e}$ and $\mathbf{A_o}$ are constructed from the $\mathbf{P}_4$ matrix in (83). Fig. 7 shows estimated power spectra of $s[n]$ before and after the application of 12^{th} and 13^{th} order distortion, and Fig. 8 shows estimated power spectra of the running sum of the quantization noise $t[n]$ before and after the application of 7^{th} and 8^{th} order distortion. As expected, the power spectra of $s^p[n]$ for $p = 1, 12$, and 13 and $t^p[n]$ for $p = 1$ and 7 show no visible spurious tones, as $o_s = 4N_t - 3 = 13$ and $o_t = 2N_t - 1 = 7$, while that of $t^8[n]$ shows spurious tones. Simulations of successive requantizers for which $N_t = 5$, with $\mathbf{A_e}$ and $\mathbf{A_o}$ as constructed from the $\mathbf{P}_5$ matrix in (84) can also be performed to corroborate that, for this case, $o_s = 4N_t - 3$ and $o_t = 2N_t - 1$ as well.

APPENDIX

Lemma 1: Suppose the conditions of Theorem 1 are satisfied. Then, for each $p \le h_t$ and each set of parity sequences $\{o_d[n],\ d = 0, 1, \ldots, K-1\}$, there exists a constant C_{t^p}, positive constants D_1, D_2, and a constant $0 < \alpha < 1$ such that for integers $n_1 \ne n_2$ (22) holds.

Proof: Without loss of generality, let $n_2 > n_1$. It follows from (7) that $t^p[n_1]t^p[n_2]$ can be expressed as

$$2^{-2pK}\left(\sum_{c_1=0}^{K-1}2^{c_1}t_{c_1}[n_1]\right)\left(\sum_{c_2=0}^{K-1}2^{c_2}t_{c_2}[n_1]\right)\cdots\left(\sum_{c_p=0}^{K-1}2^{c_p}t_{c_p}[n_1]\right)$$
$$\times\left(\sum_{d_1=0}^{K-1}2^{d_1}t_{d_1}[n_2]\right)\left(\sum_{d_2=0}^{K-1}2^{d_2}t_{d_2}[n_2]\right)\cdots\left(\sum_{d_p=0}^{K-1}2^{d_p}t_{d_p}[n_2]\right), \quad (85)$$

so $E\{t^p[n_1]t^p[n_2]\}$ can be written as

$$2^{-2pK}\sum_{c_1=0}^{K-1}\cdots\sum_{c_p=0}^{K-1}\sum_{d_1=0}^{K-1}\cdots\sum_{d_p=0}^{K-1}2^{c_1+\cdots+c_p+d_1+\cdots+d_p}$$
$$\times E\left\{\prod_{i=1}^{p}t_{d_i}[n_2]t_{c_i}[n_1]\right\}. \quad (86)$$

The above expression is a linear combination of terms of the form

$$Q(n_1,n_2) = E\left\{\prod_{j=0}^{K-1}t_j{}^{p_j}[n_1]t_j{}^{q_j}[n_2]\right\}, \quad (87)$$

where p_j and q_j are non-negative integers less than or equal to p for all $j \in \{0, 1, \ldots, K-1\}$. It thus suffices to establish a bound for $Q(n_1, n_2)$ of the form

$$|Q(n_1,n_2) - C_3| \le C_1\alpha^{n_2-n_1} + C_2\alpha^{n_1} \quad (88)$$

for some constant C_3 and some positive constants C_1 and C_2.

Equation (85) can be written in terms of conditional expectations as follows:

$$Q(n_1,n_2) = E\left\{\prod_{i=0}^{K-1}t_i{}^{p_i}[n_1]E\left\{\prod_{j=0}^{K-1}t_j{}^{q_j}[n_2]|t_d[n_1];\right.\right.$$
$$\left.\left. d = 0, 1, \ldots, K-1\right\}\right\}. \quad (89)$$

By the law of total expectation, the inner expectation in (87) can be conditioned on additional variables as long as the outer expectation in (87) is computed over all possible values of those additional variables. Thus, (87) can be rewritten as

$$Q(n_1,n_2) = E\left\{\prod_{i=0}^{K-1} t_i{}^{p_i}[n_1] E\left\{\prod_{j=0}^{K-1} t_j{}^{q_j}[n_2] | t_d[n_1], o_d[n]; d=0,1,\ldots,K-1, n=n_1+1,\ldots,n_2\right\}\right\}. \quad (90)$$

As proven in [8], the inner expectation in the right side of (88) equals

$$\prod_{j=0}^{K-1} E\left\{t_j{}^{q_j}[n_2] | t_j[n_1], o_j[n]; n=n_1+1,\ldots,n_2\right\}. \quad (91)$$

Therefore, $Q(n_1,n_2)$ can be rewritten as

$$Q(n_1,n_2) = E\left\{\prod_{i=0}^{K-1} t_i{}^{p_i}[n_1] \prod_{j=0}^{K-1} E\left\{t_j{}^{q_j}[n_2] | t_j[n_1], o_j[n]; n=n_1+1,\ldots,n_2\right\}\right\}. \quad (92)$$

By the conditions of the lemma, for each $j \in \{0,1,\ldots,K-1\}$, the vector sequence $\{\mathbf{A}_{\{o_j[n]\}}[n_1, n_2-n_1]\mathbf{t}^{(q_j)}, n_2 = n_1+1, n_1+2, \ldots\}$ converges exponentially to $b_{q_j}\mathbf{1}_{2N_t+1}$ as $n_2 - n_1 \to \infty$. Thus, there exist constants $C_{q_j} \geq 0$ and $0 < \alpha < 1$ such that

$$\left|\mathbf{A}_{\{o_j[n]\}}[n_1, n_2-n_1]\mathbf{t}^{(q_j)} - b_{q_j}\mathbf{1}_{2N_t+1}\right| \leq C_{q_j}\alpha^{n_2-n_1}\mathbf{1}_{2N_t+1}. \quad (93)$$

It follows from (8) that the ith entry of the vector $\mathbf{A}_{\{o_j[n]\}}[n_1, n_2-n_1]\mathbf{t}^{(q_j)}$ can be written as

$$\begin{aligned}\sum_{k=1}^{2N_t+1} \Pr\big(t_j[n_2] = \mathbf{t}(k) | t_j[n_1] = \mathbf{t}(i), o_j[n]; n = n_1+1,\ldots,n_2\big) \cdot \mathbf{t}^{(q_j)}(k) \\ = E\left\{t_j{}^{q_j}[n_2] | t_j[n_1] = \mathbf{t}(i), o_j[n]; n=n_1+1,\ldots,n_2\right\}.\end{aligned} \quad (94)$$

This, with (91), implies that

$$\left|E\left\{t_j{}^{q_j}[n_2] | t_j[n_1], o_j[n]; n=n_1+1,\ldots,n_2\right\} - b_{q_j}\right| \leq C_{q_j}\alpha^{n_2-n_1}. \quad (95)$$

It follows that

$$\left|\prod_{j=0}^{K-1} E\left\{t_j{}^{q_j}[n_2] | t_j[n_1], o_j[n]; n=n_1+1,\ldots,n_2\right\} - \prod_{j=0}^{K-1} b_{q_j}\right| \leq C_q\alpha^{n_2-n_1} \quad (96)$$

for some positive constant C_q.

Consider the expression

$$\left|Q(n_1,n_2) - E\left\{\prod_{j=0}^{K-1} b_{q_j} \prod_{i=0}^{K-1} t_i{}^{p_i}[n_1]\right\}\right|. \quad (97)$$

Using (90), this expression can be rewritten as

$$\left|E\left\{\prod_{i=0}^{K-1} t_i{}^{p_i}[n_1]\left(\prod_{j=0}^{K-1} E\left\{t_j{}^{q_j}[n_2] | t_j[n_1], o_j[n]; n=n_1+1,\ldots,n_2\right\} - \prod_{j=0}^{K-1} b_{q_j}\right)\right\}\right|. \quad (98)$$

Given that, for any random variable x, $|E\{x\}| \leq E\{|x|\}$, the expression in (96) is less than or equal to

$$E\left\{\prod_{i=0}^{K-1}\left|t_i{}^{p_i}[n_1]\right| \cdot \left|\prod_{j=0}^{K-1} E\left\{t_j{}^{q_j}[n_2] | t_j[n_1], o_j[n]; n=n_1+1,\ldots,n_2\right\} - \prod_{j=0}^{K-1} b_{q_j}\right|\right\}. \quad (99)$$

Since the magnitude of each $t_i[n]$ sequence is bounded by N_t and each p_i is less than or equal to p, the expression in (97) is itself less than or equal to

$$\begin{aligned}&{N_t}^{pK} E\left\{\left|\prod_{j=0}^{K-1} E\left\{t_j{}^{q_j}[n_2] | t_j[n_1], o_j[n]; n=n_1+1,\ldots,n_2\right\} - \prod_{j=0}^{K-1} b_{q_j}\right|\right\} \\ &\leq {N_t}^{pK} \max_{t_j[n_1]}\left\{\left|\prod_{j=0}^{K-1} E\left\{t_j{}^{q_j}[n_2] | t_j[n_1], o_j[n]; n=n_1+1,\ldots,n_2\right\} - \prod_{j=0}^{K-1} b_{q_j}\right|\right\}.\end{aligned} \quad (100)$$

With (94), (95)–(98) imply that

$$\left|Q(n_1,n_2) - E\left\{\prod_{j=0}^{K-1} b_{q_j} \prod_{i=0}^{K-1} t_i{}^{p_i}[n_1]\right\}\right| \leq C_1\alpha^{n_2-n_1} \quad (101)$$

for some constant $C_1 \geq 0$. By similar reasoning, it can be established that

$$\left|E\left\{\prod_{j=0}^{K-1} b_{q_j} \prod_{i=0}^{K-1} t_i{}^{p_i}[n_1]\right\} - \prod_{j=0}^{K-1} b_{q_j} \prod_{i=0}^{K-1} b_{p_i}\right| \leq C_2\alpha^{n_1} \quad (102)$$

for some constant $C_2 \geq 0$. Using the triangle inequality,

$$\begin{aligned}&\left|Q(n_1,n_2) - \prod_{j=0}^{K-1} b_{q_j} \prod_{i=0}^{K-1} b_{p_i}\right| \\ &\leq \left|Q(n_1,n_2) - E\left\{\prod_{j=0}^{K-1} b_{q_j} \prod_{i=0}^{K-1} t_i{}^{p_i}[n_1]\right\}\right| \\ &\quad + \left|E\left\{\prod_{j=0}^{K-1} b_{q_j} \prod_{i=0}^{K-1} t_i{}^{p_i}[n_1]\right\} - \prod_{j=0}^{K-1} b_{q_j} \prod_{i=0}^{K-1} b_{p_i}\right|.\end{aligned} \quad (103)$$

Therefore, it follows from (99)–(101) that (86) holds. ∎

Lemma 2: Suppose the conditions of Theorem 2 are satisfied. Then, for each $q \leq h_s$ and each set of parity sequences $\{o_d[n],\ d = 0, 1, \ldots, K-1\}$, there exists a constant C_{s^q}, positive constants E_1, E_2, and a constant $0 < \beta < 1$ such that for integers $n_1 \neq n_2$ (24) holds.

The proof is identical to that of Lemma 1 except with $s[n]$, c_{q_j}, and $\mathbf{S}_{\{o_d[n]\}}[n_1, n_2 - n_1]$ playing the roles of $t[n]$, b_{q_j}, and $\mathbf{A}_{\{o_d[n]\}}[n_1, n_2 - n_1]$, respectively. ■

Lemma 3: Suppose that $\mathbf{A_e}$ and $\mathbf{A_o}$ are centrosymmetric, that all their odd-entries row vectors contain at least $1 + \lfloor (N_t + 1)/2 \rfloor$ nonzero entries, and that all their even-entries row vectors contain at least $1 + \lfloor N_t/2 \rfloor$ nonzero entries. Then, $\mathbf{A_e}$ and $\mathbf{A_o}$ ensure order-p $t[n]$-convergence and order-q $s[n]$-convergence for all odd positive integers p and q.

Proof: Let $o_d[n]$ be any parity sequence, u be any integer, and p and q be any odd integers. It is first shown that $\mathbf{A_e}$ and $\mathbf{A_o}$ ensure order-p $t[n]$-convergence.

It follows from (9) that the elements of $\mathbf{t}^{(p)}$ satisfy

$$\mathbf{t}^{(p)}(j) = -\mathbf{t}^{(p)}(2N_t + 2 - j). \tag{104}$$

For any integers m_1 and $m_2 > 0$, (10) implies that $\mathbf{A}_{\{o_d[n]\}}[m_1, m_2]$ is either a one-step state transition matrix or can be expanded as a product of such matrices. Since each such matrix equals $\mathbf{A_e}$ and $\mathbf{A_o}$, the conditions of the lemma imply that $\mathbf{A}_{\{o_d[n]\}}[m_1, m_2]$ is centrosymmetric or can be expanded as a product of centrosymmetric matrices. Therefore, by the properties of centrosymmetric matrices, $\mathbf{A}_{\{o_d[n]\}}[m_1, m_2]$ is centrosymmetric [19]. This, together with (102), implies that the elements of $\mathbf{A}_{\{o_d[n]\}}[m_1,\ m_2]\mathbf{t}^{(p)}$ satisfy (105), where the last expression in (105) can be written as

$$-\sum_{j=1}^{2N_t+1} \left(\mathbf{A}_{\{o_d[n]\}}[m_1, m_2]\right)(2N_t + 2 - i, j) \cdot \mathbf{t}^{(p)}(j). \tag{106}$$

Thus, the elements of $\mathbf{A}_{\{o_d[n]\}}[m_1, m_2]\mathbf{t}^{(p)}$ satisfy

$$\begin{aligned}\left(\mathbf{A}_{\{o_d[n]\}}[m_1, m_2]\mathbf{t}^{(p)}\right)(i) \\ = -\left(\mathbf{A}_{\{o_d[n]\}}[m_1, m_2]\mathbf{t}^{(p)}\right)(2N_t + 2 - i).\end{aligned} \tag{107}$$

Let m be any integer greater than 1. Given (10), $\mathbf{A}_{\{o_d[n]\}}[u, m]\mathbf{t}^{(p)}$ can be written as

$$\mathbf{A}_{\{o_d[n]\}}[u, 1]\mathbf{A}_{\{o_d[n]\}}[u + 1, m - 1]\mathbf{t}^{(p)}. \tag{108}$$

Therefore, the ith element of $\mathbf{A}_{\{o_d[n]\}}[u, m]\mathbf{t}^{(p)}$ equals

$$\begin{aligned}&\left(\mathbf{A}_{\{o_d[n]\}}[u, m]\mathbf{t}^{(p)}\right)(i) = \\ &\sum_{j=1}^{2N_t+1} \left(\mathbf{A}_{\{o_d[n]\}}[u, 1]\right)(i, j) \cdot \left(\mathbf{A}_{\{o_d[n]\}}[u+1, m-1]\mathbf{t}^{(p)}\right)(j).\end{aligned} \tag{109}$$

Let $(A_{i,1}, A_{i,2}, \ldots, A_{i,2N_t+1})$ be a permutation of

$$\left(\left(\mathbf{A}_{o_d[n]}[u, 1]\right)(i, j), j = 1, 2, \ldots, 2N_t + 1\right) \tag{110}$$

which satisfies

$$A_{i,1} \geq A_{i,2} \geq \ldots \geq A_{i,2N_t+1} \tag{111}$$

and $(t_1, t_2, \ldots, t_{2N_t+1})$ a permutation of

$$\left(\left(\mathbf{A}_{o_d[n]}[u + 1, m - 1]\mathbf{t}^{(p)}\right)(j), j = 1, 2, \ldots, 2N_t + 1\right) \tag{112}$$

which satisfies

$$t_1 \geq t_2 \geq \ldots \geq t_{2N_t+1}. \tag{113}$$

Then, it follows from (107) and from the rearrangement inequality that

$$\left(\mathbf{A}_{\{o_d[n]\}}[u, m]\mathbf{t}^{(p)}\right)(i) \leq \sum_{j=1}^{2N_t+1} A_{i,j}t_j. \tag{114}$$

Given (105) and that the elements of $\mathbf{A}_{\{o_d[n]\}}[u, 1]$ are all nonnegative, $A_{i,j}t_j \leq 0$ for $j \in \{N_t + 1, N_t + 2, \ldots, 2N_t + 1\}$. Thus, (112) implies that

$$\left(\mathbf{A}_{\{o_d[n]\}}[u, m]\mathbf{t}^{(p)}\right)(i) \leq \sum_{j=1}^{N_t} A_{i,j}t_j \leq t_1 \sum_{j=1}^{N_t} A_{i,j}. \tag{115}$$

It is now shown that

$$\sum_{j=1}^{N_t} A_{i,j} < \sum_{j=1}^{2N_t+1} A_{i,j} = 1. \tag{116}$$

Since $\mathbf{A}_{\{o_d[n]\}}[u, 1]$ is stochastic, all of its elements are nonnegative, and the sum of the elements in each of its rows is 1. Additionally, since it equals either $\mathbf{A_e}$ or $\mathbf{A_o}$, $\mathbf{A}_{\{o_d[n]\}}[u, 1]$ is either an even-entries or an odd-entries matrix, so its row vectors alternate between even-entries and odd-entries vectors. Suppose N_t is even. If the ith row of $\mathbf{A}_{\{o_d[n]\}}[u, 1]$ is an even-entries row then $A_{i,j} = 0$ for $j \in \{1, 3, \ldots, 2N_t + 1\}$. By the conditions of

$$\begin{aligned}\left(\mathbf{A}_{\{o_d[n]\}}[m_1, m_2]\mathbf{t}^{(p)}\right)(i) &= \sum_{j=1}^{2N_t+1} \left(\mathbf{A}_{\{o_d[n]\}}[m_1, m_2]\right)(i, j)\mathbf{t}^{(p)}(j) = \\ -\sum_{j=1}^{2N_t+1} &\left(\mathbf{A}_{\{o_d[n]\}}[m_1, m_2]\right)(2N_t + 2 - i, 2N_t + 2 - j)\mathbf{t}^{(p)}(2N_t + 2 - j),\end{aligned} \tag{105}$$

the lemma, $A_{i,j}$ is nonzero for at least $1+N_t/2$ values of $j \in \{2,4,\ldots,N_t,N_t+2,\ldots,2N_t\}$, so it is nonzero for at least 1 value of $j \in \{N_t+2,N_t+4,\ldots,2N_t\}$, which implies that (114) holds. Similarly, if the ith row of $\mathbf{A}_{\{o_d[n]\}}[u,1]$ is an odd-entries row then $A_{i,j}=0$ for $j \in \{2,4,\ldots,2N_t\}$. By the conditions of the lemma, $A_{i,j}$ is nonzero for at least $1+N_t/2$ values of $j \in \{1,3,\ldots,N_t-1,N_t+1,\ldots,2N_t+1\}$, so it is nonzero for at least one value of $j \in \{N_t+1,N_t+3,\ldots,2N_t+1\}$, which implies that (114) holds. Now suppose N_t is odd. If the ith row of $\mathbf{A}_{\{o_d[n]\}}[u,1]$ is an even-entries row then $A_{i,j}=0$ for $j \in \{1,3,\ldots,2N_t+1\}$. By the conditions of the lemma, $A_{i,j}$ is nonzero for at least $1+(N_t-1)/2$ values of $j \in \{2,4,\ldots,N_t-1,N_t+1,\ldots,2N_t\}$, so it is nonzero for at least 1 value of $j \in \{N_t+1,N_t+3,\ldots,2N_t\}$, which implies that (114) holds. Similarly, if the ith row of $\mathbf{A}_{\{o_d[n]\}}[u,1]$ is an odd-entries row then $A_{i,j}=0$ for $j \in \{2,4,\ldots,2N_t\}$. By the conditions of the lemma, $A_{i,j}$ is nonzero for at least $1+(N_t+1)/2$ values of $j \in \{1,3,\ldots,N_t,N_t+2,\ldots,2N_t+1\}$, so it is nonzero for at least 1 value of $j \in \{N_t+2,N_t+4,\ldots,2N_t+1\}$, which implies that (114) holds.

It follows from (113) and (114) that

$$\max_i\left\{\left(\mathbf{A}_{\{o_d[n]\}}[u,m]\mathbf{t}^{(p)}\right)(i)\right\} \le (1-\beta)\max_i\left\{\left(\mathbf{A}_{\{o_d[n]\}}[u+1,m-1]\mathbf{t}^{(p)}\right)(i)\right\}, \quad (117)$$

where β is the smallest nonzero element of $\mathbf{A_e}$ and $\mathbf{A_o}$, i.e.,

$$\beta=\min\left\{\min_{i,j}\left\{\mathbf{A_e}(i,j)\neq 0\right\},\min_{i,j}\left\{\mathbf{A_o}(i,j)\neq 0\right\}\right\}. \quad (118)$$

Since (115) holds for any integers u and $m>1$, it follows that, for $m>2$,

$$\begin{aligned}\max_i\left\{\left(\mathbf{A}_{\{o_d[n]\}}[u,m]\mathbf{t}^{(p)}\right)(i)\right\} &\le (1-\beta)^2\max_i\left\{\left(\mathbf{A}_{\{o_d[n]\}}[u+2,m-2]\mathbf{t}^{(p)}\right)(i)\right\} \le \cdots \\ &\le (1-\beta)^{m-1}\max_i\left\{\left(\mathbf{A}_{\{o_d[n]\}}[u+m-1,1]\mathbf{t}^{(p)}\right)(i)\right\} \\ &\le (1-\beta)^{m-1}\max_i\left\{\mathbf{t}^{(p)}(i)\right\}, \end{aligned} \quad (119)$$

where the last inequality holds because $\mathbf{A}_{\{o_d[n]\}}[u+m-1,1]$ is stochastic, so all of its elements are nonnegative and the sum of the elements in each of its rows is 1. Thus,

$$\lim_{m\to\infty}\left[\max_i\left\{\left(\mathbf{A}_{\{o_d[n]\}}[u,m]\mathbf{t}^{(p)}\right)(i)\right\}\right]=0 \quad (120)$$

with exponential convergence.

Note now that (105) implies that

$$\min_i\left\{\left(\mathbf{A}_{\{o_d[n]\}}[u,m]\mathbf{t}^{(p)}\right)(i)\right\} = -\max_i\left\{\left(\mathbf{A}_{\{o_d[n]\}}[u,m]\mathbf{t}^{(p)}\right)(i)\right\} \quad (121)$$

for each m, so

$$\lim_{m\to\infty}\left[\min_i\left\{\left(\mathbf{A}_{\{o_d[n]\}}[u,m]\mathbf{t}^{(p)}\right)(i)\right\}\right]=0 \quad (122)$$

with exponential convergence as well. Hence, (118), (120), and the squeeze theorem from calculus imply that $\mathbf{A_e}$ and $\mathbf{A_o}$ ensure order-p $t[n]$-convergence.

The proof that $\mathbf{A_e}$ and $\mathbf{A_o}$ ensure order-q $s[n]$-convergence is similar. It follows from (15) that the elements of $\mathbf{s}^{(q)}$ satisfy

$$\mathbf{s}^{(q)}(j)=-\mathbf{s}^{(q)}(4N_t+2-j). \quad (123)$$

Given that $\mathbf{A_e}$ and $\mathbf{A_o}$ are centrosymmetric, (17) implies that $\mathbf{S_e}$ and $\mathbf{S_o}$ are also centrosymmetric. For any integers m_1 and $m_2>0$, (10), (16), and the conditions of the lemma imply that $\mathbf{S}_{\{o_d[n]\}}[m_1,m_2]$ is centrosymmetric or can be expanded as a product of centrosymmetric matrices. Therefore, $\mathbf{S}_{\{o_d[n]\}}[m_1,m_2]$ is centrosymmetric. This, together with (121), implies that the elements of $\mathbf{S}_{\{o_d[n]\}}[m_1,m_2]\mathbf{s}^{(q)}$ satisfy

$$\left(\mathbf{S}_{\{o_d[n]\}}[m_1,m_2]\mathbf{s}^{(q)}\right)(i) = -\left(\mathbf{S}_{\{o_d[n]\}}[m_1,m_2]\mathbf{s}^{(q)}\right)(2N_t+2-i). \quad (124)$$

The rest of the proof follows from reasoning similar to that presented above from (105) to (120). ∎

Lemma 4: Let

$$\mathbf{A_e}\mathbf{t}^{(p)}=\mathbf{A_o}\mathbf{t}^{(p)}=b_p\mathbf{1}_{2N_t+1} \quad (125)$$

for some integer p and some constant b_p. Then, $\mathbf{A_e}$ and $\mathbf{A_o}$ ensure order-p $t[n]$-convergence.

Proof: Let $o_d[n]$ be any parity sequence and u be any integer. It follows from (10) that for each integer $m>1$ $\mathbf{A}_{\{o_d[n]\}}[u,m]$ can be written as

$$\mathbf{A}_{\{o_d[n]\}}[u,m]=\mathbf{A}_{\{o_d[n]\}}[u,m-1]\mathbf{A}_{\{o_d[n]\}}[u+m-1,1]. \quad (126)$$

Since $\mathbf{A}_{\{o_d[n]\}}[u+m-1,1]$ equals either $\mathbf{A_e}$ or $\mathbf{A_o}$, (123) and (124) imply that

$$\mathbf{A}_{\{o_d[n]\}}[u,m]\mathbf{t}^{(p)}=b_p\mathbf{A}_{\{o_d[n]\}}[u,m-1]\mathbf{1}_{2N_t+1}. \quad (127)$$

Additionally, $\mathbf{A}_{\{o_d[n]\}}[u,m-1]$ is stochastic, so

$$\mathbf{A}_{\{o_d[n]\}}[u,m-1]\mathbf{1}_{2N_t+1}=\mathbf{1}_{2N_t+1}. \quad (128)$$

Therefore,

$$\mathbf{A}_{\{o_d[n]\}}[u,m]\mathbf{t}^{(p)}=b_p\mathbf{1}_{2N_t+1}, \quad (129)$$

so for each integer $m\ge 1$

$$\left|\mathbf{A}_{\{o_d[n]\}}[u,m]\mathbf{t}^{(p)}-b_p\mathbf{1}_{2N_t+1}\right| \le \mathbf{0}_{2N_t+1}, \quad (130)$$

which proves the lemma. ∎

Lemma 5: Given any parity sequence $o_d[n]$ and any integers u and $m>0$, $\mathbf{A}_{\{o_d[n]\}}[u,m]$ is either an even-entries or an odd-entries matrix.

Proof: It follows from (11) that

$$\sum_{k=u+1}^{u+m}s_d[k]=\sum_{k=u+1}^{u+m}\left(t_d[k]-t_d[k-1]\right) = t_d[u+m]-t_d[u]. \quad (131)$$

Since $s_d[n]$ and $o_d[n]$ have the same parity at each n, the parity of each $s_d[k]$, for $k = u+1, u+2, \ldots, u+m$, is fixed, i.e., each $s_d[k]$ is either even or odd. Therefore, the parity of the sum in the left side of (129) is also fixed. If the sum is even, (129) implies that the probability that $t_d[u+m]$ and $t_d[u]$ have different parities is zero. Similarly, if it is odd, the probability that $t_d[u+m]$ and $t_d[u]$ have the same parity is zero. Therefore, (8) implies that $\mathbf{A}_{\{o_d[n]\}}[u,m]$ is either an even-entries or an odd-entries matrix. ■

Lemma 6: Let (79) hold for some integer q and some constant c_q. Then, $\mathbf{A_e}$ and $\mathbf{A_o}$ ensure order-q $s[n]$-convergence.

Proof: Let $o_d[n]$ be any parity sequence and u be any integer. It follows from (10) and (16) that for each integer $m > 2$ $\mathbf{S}_{\{o_d[n]\}}[u,m]$ can be written as

$$\mathbf{A}_{\{o_d[n]\}}[u,m-2]\mathbf{A}_{\{o_d[n]\}}[u+m-2,1]\mathbf{S}_{\{o_d[n]\}}[u+m-1,1]. \tag{132}$$

The rest of the proof is almost identical to the proof of Lemma 4, with $\mathbf{A}_{\{o_d[n]\}}[u+m-2,1]\mathbf{S}_{\{o_d[n]\}}[u+m-1,1]$, $\mathbf{s}^{(q)}$, and c_q playing the roles of $\mathbf{A}_{\{o_d[n]\}}[u+m-1,1]$, $\mathbf{t}^{(p)}$, and b_p, respectively. ■

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