Conversion Error in D/A Converters Employing Dynamic Element Matching

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ABSTRACT

An approach to reducing harmonic distortion in D/A converters involves a technique called dynamic element matching. The idea is not so much to reduce the power of overall conversion error but rather to give it a random, noise-like structure. This paper presents the first rigorous theoretical analysis of the conversion error introduced by an important class of D/A converters based on dynamic element matching. Specifically, the paper develops the rather surprising result that, under mild circuit performance assumptions, the only nonlinear distortion introduced by this class of D/A converters is that of additive white noise with a DC offset.

INTRODUCTION

Ideally, a digital-to-analog (D/A) converter transforms a sequence of values represented as b-bit numbers to exactly the same sequence of values represented as analog voltages. Consequently, from a signal processing point of view, an ideal D/A converter is a linear system. However, practical D/A converters introduce errors that cause the values represented as analog voltages to differ from the corresponding values represented as b-bit numbers. Thus, the sequence of output values can be written as $y(n) = x(n) + \epsilon(n)$ where $x(n)$ is the sequence of input values and $\epsilon(n)$ is a sequence representing the D/A conversion error. In general, $\epsilon(n)$ is a non-linear function of the input sequence, so practical D/A converters introduce harmonic distortion.

An approach to reducing harmonic distortion in D/A converters involves a technique called *dynamic element matching* [1]. The idea is not so much to reduce the power of $\epsilon(n)$, but rather to give it a random, noise-like structure. By reducing the correlation among successive samples of $\epsilon(n)$, harmonic distortion is reduced. A particular version of this approach, introduced by Carley [1], is an extension of the well known flash technique often used for fast A/D conversion. For lack of an existing name, this paper will refer to D/A converters based on Carley's approach as *stochastic flash* D/A converters. The main contribution of this paper is a rigorous theoretical analysis of the conversion error introduced by stochastic flash D/A converters.

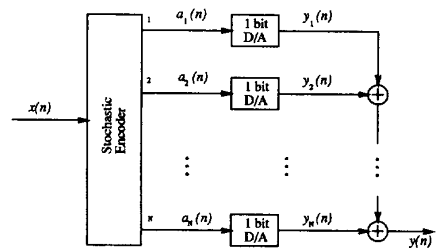

Figure 1: The stochastic flash D/A converter.

THE D/A CONVERTER ARCHITECTURE

The stochastic flash D/A converter architecture is shown in Figure 1. It operates on a sequence of digital numbers, $x(n)$, and produces a sequence of analog output values, $y(n)$. In a manner that will be described below, the *stochastic encoder* maps each input value to N one-bit output values, $a_1(n), \ldots, a_N(n)$. These are converted into analog voltages or currents by the N one-bit D/A converters and then summed to produce $y(n)$.

The input sequence, $x(n)$, is assumed to be a deterministic sequence of numbers with values $x(n) \in \{x_{min} + k\Delta : k = 0, 1, \ldots, N\}$ where $N > 1$. Thus, $x_{min} \leq x(n) \leq x_{max}$ where $x_{max} = x_{min} + N\Delta$. For each value of $x(n)$, the stochastic encoder sets $K(n)$ of its output lines to one and the remaining $N - K(n)$ of its output lines to zero where

$$K(n) = [x(n) - x_{min}]/\Delta. \qquad (1)$$

For each value of $x(n)$, this can be done in $N_k = \binom{N}{K(n)}$ different ways. The idea behind dynamic element matching is to have the stochastic encoder choose among the N_k possibilities randomly. Formally, the operation of the stochastic encoder can be described as follows. Let $\mathbf{a}(n) = [a_1(n) \cdots a_N(n)]^T$. Then the stochastic encoder chooses its outputs such that

$$\mathbf{a}(n) \in \{0,1\}^N \quad \text{and} \quad \mathbf{a}^T(n)\mathbf{a}(n) = K(n), \qquad (2)$$

where $\mathbf{a}(n)$ is a sequence of independent random variables such that for every value of $x(n)$ each of the N_k possible values of $\mathbf{a}(n)$ are equiprobable. The one-bit D/A converters each operate according to

$$y_r(n) = \begin{cases} w_h + e_{h_r} & \text{if } a_r(n) = 1; \\ w_l + e_{h_l} & \text{if } a_r(n) = 0; \end{cases}$$

where $w_h = x_{max}/N$, $w_l = x_{min}/N$ (hence, $w_h - w_l = \Delta$), and e_{h_r}, e_{l_r} represent D/A conversion errors. If the one-bit D/A converters are ideal, $e_{hr} = e_{lr} = 0$ for all r, and

$$y(n) = \mathbf{a}^T(n)[w_h \dots w_h]^T + \bar{\mathbf{a}}^T(n)[w_l \dots w_l]^T,$$

where $\bar{\mathbf{a}}^T(n)$ represents the one's-complement of $\mathbf{a}^T(n)$. It can be easily shown that, in this case, $y(n) = x(n)$.

Now suppose the one-bit D/A converters are not ideal. Then, $\mathbf{e}_h \neq \mathbf{0}$ or $\mathbf{e}_l \neq \mathbf{0}$ where $\mathbf{e}_h = [e_{h1} \dots e_{hN}]^T$ and $\mathbf{e}_l = [e_{l1} \dots e_{lN}]^T$. In this case, the output $y(n)$ can be written as

$$y(n) = \mathbf{a}^T(n)\left([w_h \dots w_h]^T + [e_{h1} \dots e_{hN}]^T\right) + \bar{\mathbf{a}}^T(n)\left([w_l \dots w_l]^T + [e_{l1} \dots e_{lN}]^T\right),$$

which can be simplified as

$$y(n) = x(n) + \mathbf{a}^T(n)\mathbf{e}_h + \bar{\mathbf{a}}^T(n)\mathbf{e}_l. \tag{3}$$

Thus the conversion error introduced by the non-ideal one-bit D/A converters is $\epsilon(n) = \mathbf{a}^T(n)\mathbf{e}_h + \bar{\mathbf{a}}^T(n)\mathbf{e}_l$.

SECOND-ORDER STATISTICS OF THE D/A CONVERTER

It will now be shown that provided the one-bit D/A converter level errors do not change over time, the randomization performed by the stochastic encoder decorrelates $\epsilon(n)$ such that harmonic distortion is eliminated. In accordance with the usual definitions, the time-average autocorrelation and mean of the input sequence $x(n)$ are defined as: $\overline{R}_{xx}(k) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} x(n)x(n+k)$, and $\overline{M}_x = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} x(n)$ respectively. The time-average autocorrelation and mean of the output sequence are defined analogously. The following theorem asserts that with probability 1 the time-average autocorrelation of $y(n)$ is the linear combination of the time-average autocorrelation of $x(n)$, a constant, and a white noise term. Thus, instead of introducing harmonic distortion, the stochastic flash D/A converter introduces white noise with a DC component.

Theorem: If $\overline{M}_x$ and $\overline{R}_{xx}(k)$ exist and e_{h_r} and e_{l_r} are not functions of time, then

$$\overline{M}_y = \overline{M}_x(1 + c_0) + c_1, \tag{4}$$

and

$$\overline{R}_{yy}(k) = (1+\alpha)\overline{R}_{xx}(k) + \overline{\beta} + \overline{\sigma}^2\delta(k), \tag{5}$$

with probability 1, where the function $\delta(\cdot)$ is the Kronecker delta, the symbol $\mathbf{1}^T$ is defined as the all-ones vector: $[1\cdots1]$ and

$$c_0 = \frac{(\mathbf{1}^T\mathbf{e}_h - \mathbf{1}^T\mathbf{e}_l)}{N\Delta}, \quad c_1 = \frac{(x_{min}\mathbf{1}^T\mathbf{e}_h - x_{max}\mathbf{1}^T\mathbf{e})}{N\Delta},$$

$$\alpha = c_0(2 + c_0),\ \overline{\beta} = 2c_1(1 + c_0)\overline{M}_x + c_1^2,$$

$$\overline{\sigma}^2 = \frac{1}{(N-1)}\left[c_0^2 - \frac{(\mathbf{e}_h - \mathbf{e}_l)^T(\mathbf{e}_h - \mathbf{e}_l)}{N\Delta^2}\right] \cdot \left[x_{min}x_{max} + \overline{R}_{xx}(0) - (x_{min} + x_{max})\overline{M}_x\right].$$

Proof: Only the proof of (5) will be presented, because the proof of (4) is similar. Consider $R_{yy}(n,k)$, the statistical autocorrelation of $y(n)$ defined as $R_{yy}(n,k) = \mathrm{E}\big[y(n)y(n+k)\big]$. Because $x(n)$ is deterministic and $\mathbf{a}(n)$ is a sequence of independent random variables, it follows from (3) that $y(n)$ is also a sequence of independent random variables. Therefore,

$$R_{yy}(n,k) = \mathrm{E}\big[y(n)\big]\,\mathrm{E}\big[y(n+k)\big] + \sigma^2(n)\delta(k),$$

where $\sigma^2(n) = \mathrm{E}\big[y^2(n)\big] - \Big[\mathrm{E}\big[y(n)\big]\Big]^2$.

From (1), (3) and the lemma presented in the Appendix it follows that

$$E[y(n)] = x(n)(1 + c_0) + c_1. \tag{6}$$

Applying (6) and the definition of α gives

$$R_{yy}(n,k) = x(n)x(n+k)(1+\alpha) + \big[x(n) + x(n+k)\big]c_1(1 + c_0) + c_1^2 + \sigma^2(n)\delta(k). \tag{7}$$

From (3), it follows that

$$\mathrm{E}\big[y^2(n)\big] = x^2(n) + 2x(n)\,\mathrm{E}\big[\mathbf{a}^T(n)\mathbf{e}_h + \bar{\mathbf{a}}^T(n)\mathbf{e}_l\big] + \mathrm{E}\big[\mathbf{a}^T(n)\mathbf{e}_h + \bar{\mathbf{a}}^T(n)\mathbf{e}_l\big]^2. \tag{8}$$

It is convenient to evaluate the second two terms in (8) separately. Applying the first equation of the lemma presented in the appendix, the second term becomes

$$\mathrm{E}\big[\mathbf{a}^T(n)\mathbf{e}_h + \bar{\mathbf{a}}^T(n)\mathbf{e}_l\big] = \frac{K(n)}{N}\mathbf{1}^T\mathbf{e}_h + \frac{N - K(n)}{N}\mathbf{1}^T\mathbf{e}_l. \tag{9}$$

With $\mathbf{A}_N$ defined as the $N \times N$ all-ones matrix and $\mathbf{I}_N$ defined as the $N \times N$ identity matrix, note that for any N-length vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a}^T\mathbf{A}_N\mathbf{b} = (\mathbf{1}^T\mathbf{a})(\mathbf{1}^T\mathbf{b})$, and

$\mathbf{a}^T\mathbf{I}_N\mathbf{b} = \mathbf{a}^T\mathbf{b}$. Using these relationships and the results of the Lemma presented in the appendix gives

$$\begin{aligned} \mathrm{E}\big[\mathbf{a}^T(n)\mathbf{e}_h + \overline{\mathbf{a}}^T(n)\mathbf{e}_l\big]^2 &= \frac{K(n)[K(n)-1]}{N(N-1)}(\mathbf{1}^T\mathbf{e}_h)^2 \\ &+ \frac{K(n)[N-K(n)]}{N(N-1)}\mathbf{e}_h^T\mathbf{e}_h + \frac{K(n)[N-K(n)]}{N(N-1)}\mathbf{e}_l^T\mathbf{e}_l \\ &+ \frac{[N-K(n)][N-K(n)-1]}{N(N-1)}(\mathbf{1}^T\mathbf{e}_l)^2 \\ &+ 2\frac{K(n)[N-K(n)]}{N(N-1)}\big[(\mathbf{1}^T\mathbf{e}_h)(\mathbf{1}^T\mathbf{e}_l) - \mathbf{e}_h^T\mathbf{e}_l\big]. \end{aligned} \tag{10}$$

Substituting (9) and (10) into (8), collecting terms, using the identity: $\mathbf{e}_h^T\mathbf{e}_h + \mathbf{e}_l^T\mathbf{e}_l - 2\mathbf{e}_h^T\mathbf{e}_l = (\mathbf{e}_h - \mathbf{e}_l)^T(\mathbf{e}_h - \mathbf{e}_l)$ and expanding $K(n)$ using (1) gives

$$\sigma^2(n) = \frac{1}{(N-1)}\left[c_0^2 - \frac{(\mathbf{e}_h - \mathbf{e}_l)^T(\mathbf{e}_h - \mathbf{e}_l)}{N\Delta^2}\right] \cdot \left[x_{min}x_{max} + x^2(n) - (x_{min} + x_{max})x(n)\right]. \tag{11}$$

It follows from (7), (11), and the definitions of $\overline{\beta}$ and $\overline{\sigma}^2$ that $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} R_{yy}(n,k) = (1+\alpha)\overline{R}_{xx}(k) + \overline{\beta} + \overline{\sigma}^2\delta(k)$. Therefore, to finish the proof it is sufficient to show that

$$\overline{R}_{yy}(k) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} R_{yy}(n,k) \tag{12}$$

with probability 1, or, equivalently, that for each k the sequence $y(n)y(n+k)$ obeys the strong law of large numbers. Because of symmetry, it is sufficient to prove the result for $k \geq 0$. Choose any fixed $k \geq 0$. For each $p = 0,\ldots,k-1$, consider the subsequence $r_p(n) = y(n(k+1)+p)y(n(k+1)+p+k)$. Because $y(n)$ is a sequence of independent random variables, it follows that, for each p, $r_p(n)$ is a sequence of independent random variables. Moreover, since $y(n)$ is a bounded sequence, $r_p(n)$ must also be a bounded sequence. Therefore, there exists some number C' such that $\mathrm{Var}[r_p(n)] < C'$ for all n and all p. It follows from the Kolmogorov Criterion [2] that for each p, $r_p(n)$ obeys the strong law of large numbers.

Since, by definition, $\mathrm{E}[r_p(n)] = R_{yy}(n(k+1)+p,k)$, $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} r_p(n) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} R_{yy}(n(k+1)+p,k)$ with probability 1. This holds for $p = 0,\ldots,k-1$, so it follows that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\sum_{p=0}^{k-1} r_p(n) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\sum_{p=0}^{k-1} R_{yy}(n(k+1)+p,k)$$

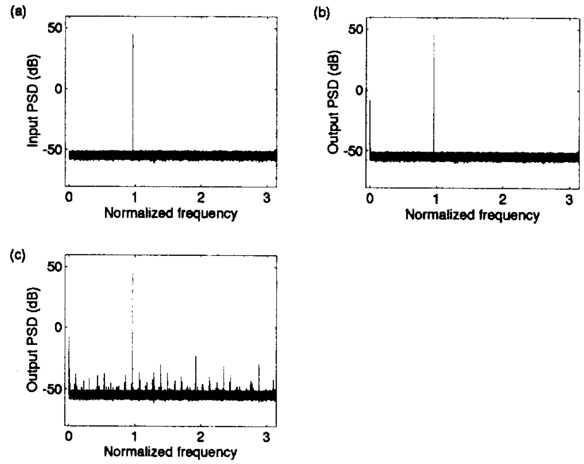

Figure 2: Estimated power spectral density (PSD) in dB relative to the power of a full scale DC input of (a) the input sequence, (b) the stochastic flash D/A converter output sequence, and (c) the deterministic flash D/A converter output sequence.

with probability 1, which is equivalent to (12). ■

SIMULATION RESULTS

This section presents simulation data that support the theoretical results derived above. As will be shown, no harmonic distortion is evident in the simulated output sequence of an 8-bit stochastic flash D/A converter with a dithered sinusoidal input. The simulations used an input sequence that was generated by adding a dither sequence to a sinusoid and then quantizing the result to 8-bits. The sinusoid had an amplitude of 127Δ, where Δ is the value of the least-significant-bit of the quantizer. The dither sequence was an independent identically distributed sequence with a triangular probability density function supported on $(-\Delta,\Delta)$. Thus, by design the input sequence behaved as an ideal sinusoid plus white noise [3].

Figure 2a shows the estimated power spectral density of the dithered sinusoid input sequence. The data were obtained by averaging 15 periodograms each corresponding to $2^{19} = 524288$ points of the dithered sinusoid input sequence. The frequency of the sinusoid was set to $160000\pi/524288$ radians so as to avoid spectral leakage. Figure 2b shows the estimated power spectral density of the output of an 8-bit stochastic flash D/A converter operating on the dithered sinusoidal input sequence. The one-bit D/A converter errors, e_{h_r} and e_{l_r}, $1 \leq r \leq 256$, were chosen randomly within a range of $\pm 2.5\%$ of Δ. As predicted by the theory developed in the previous section, no harmonic distortion is evident in the figure. For comparison, Figure 2c shows the estimated power spectral density of the output of a version of the D/A converter in which no encoder randomization is performed.

CONCLUSION

It has been shown in this paper that, in principle, the stochastic flash D/A converter introduces no harmonic distortion whatsoever. In particular, no matter how much error is introduced by the one-bit D/A converters within the stochastic flash D/A converter or how these errors are distributed, the system behaves exactly like an ideal D/A converter aside from a white noise term and gain and DC offsets.

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APPENDIX

Lemma: Suppose that $K(n)$ is an integer-valued function such that $0 \leq K(n) \leq N$, and assume $\mathbf{a}(n) = [a_1 \cdots a_N]^T$ as a sequence of independent random variables satisfying $\mathbf{a}(n) \in \{0,1\}^N$, and $\mathbf{a}^T(n)\mathbf{a}(n) = K(n)$, where for every given value of $K(n)$ each of the $N_k = \binom{N}{K(n)}$ possible values of $\mathbf{a}(n)$ are equiprobable. Then, for $N > 1$,

$$\mathrm{E}\big[\mathbf{a}(n)\big] = \frac{K(n)}{N}[1\ldots 1]^T, \tag{13}$$

$$\mathrm{E}\big[\mathbf{a}(n)\mathbf{a}^T(n)\big] = \frac{K(n)[K(n)-1]}{N(N-1)}\mathbf{A_N} + \frac{K(n)[N-K(n)]}{N(N-1)}\mathbf{I_N}, \tag{14}$$

$$\mathrm{E}\big[\mathbf{a}(n)\overline{\mathbf{a}}^T(n)\big] = \frac{K(n)[N-K(n)]}{N(N-1)}(\mathbf{A_N} - \mathbf{I_N}), \tag{15}$$

where $\mathbf{I_N}$ represents the $N \times N$ identity matrix and $\mathbf{A_N}$ the $N \times N$ all-ones matrix.

Proof: Let N_K be the number of different vectors $\mathbf{a}(n)$ such that the sum of their elements is equal to $K(n)$. By definition $K(n) \in \{0,\ldots,N\}$, so $N_K = \binom{N}{K(n)}$. From the definition of the expectation operator and the equiprobability of the different values of $\mathbf{a}(n)$, $\mathrm{E}\big[\mathbf{a}(n)\big] = \frac{1}{N_K}\sum_{i=1}^{N_K}\mathbf{a}^{(i)}(n)$ where $\mathbf{a}^{(i)}(n)$ represents the i-th vector $\mathbf{a}(n)$ that obeys the constraint. Suppose $K(n) \neq 0$. Consider the r-th element, $a_r^{(i)}(n)$, of $\mathbf{a}^{(i)}(n)$. The number of distinct values of i for which $a_r^{(i)}(n)$ is equal to 1 corresponds to the number of times the sum of the $N-1$ elements with index different from r is equal to $K(n)-1$, i.e., $\binom{N-1}{K(n)-1}$. Since this reasoning applies for each $0 \leq r \leq N$, $\mathrm{E}\big[\mathbf{a}(n)\big] = \frac{K(n)}{N}[1\ldots 1]^T$. If $K(n) = 0$ then $N_K = 1$, $\mathbf{a}^T(n) = [0\cdots 0]$ and (13) follows trivially.

By exploiting the definition and the the equiprobability of the vectors $\mathbf{a}^{(i)}(n)$,

$$\mathrm{E}\big[\mathbf{a}(n)\mathbf{a}^T(n)\big] = \frac{1}{N_K}\sum_{i=1}^{N_K}\mathbf{a}^{(i)}(n)\mathbf{a}^{(i)T}(n).$$

The principal diagonal of the matrix $\mathbf{a}(n)\mathbf{a}^T(n)$ is equal to $\mathbf{a}(n)$ and, therefore, by the reasoning employed in the proof of (13), it follows that

$$\mathrm{diag}\Big\{\frac{1}{N_K}\sum_{i=1}^{N_K}\mathbf{a}^{(i)}(n)\mathbf{a}^{(i)T}(n)\Big\} = \frac{K(n)}{N}[1\ldots 1]^T. \tag{16}$$

Consider an off-diagonal element $a_{j,k}^{(i)}$ of matrix $\mathbf{a}^{(i)}(n)$ $\mathbf{a}^{(i)T}(n)$. Assume $K(n) > 1$. Consider the number of times $a_{j,k}^{(i)} = 1$ as $i = 1,\ldots,N_K$. If the element $a_{k,k}^{(i)}$ on the principal diagonal is equal to zero, then $a_{j,k}^{(i)} = 0$. Therefore, the number of times $a_{j,k}^{(i)} = 1$ and $a_{k,k}^{(i)} = 1$, is equal to the number of times the sum of the $N-2$ elements of column k with row index different from j and from k, is equal to $K(n)-2$, i.e., $\binom{N-2}{K(n)-2}$. Therefore, the expectation of each off-diagonal element is equal to

$$\frac{1}{N_K}\sum_{i=1}^{N_K}\mathbf{a}_i(n)\mathbf{a}_i^T(n)(\mathbf{A_N} - \mathbf{I_N}) = \binom{N-2}{K(n)-2}\cdot \binom{N}{K(n)}^{-1}(\mathbf{A_N} - \mathbf{I_N}) = \frac{K(n)[K(n)-1]}{N(N-1)}(\mathbf{A_N} - \mathbf{I_N}). \tag{17}$$

By combining (16) and (17), (14) follows for $K(n) > 1$. If $K(n) = 0$ then $\mathrm{E}\big[\mathbf{a}(n)\mathbf{a}^T(n)\big]$ equals the all-zeros matrix and (14) trivially follows. If $K(n) = 1$ then the off-diagonal elements of matrix $\mathbf{a}(n)\mathbf{a}^T(n)$ are equal to zero. As a consequence $\mathrm{E}\big[\mathbf{a}^T(n)\big] = \frac{K(n)}{N}\mathbf{I_N}$ and (14) again follows.

The proof of (15) is similar to that of (14).

■

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